ORDER PRESERVATION IN LIMIT ALGEBRAS

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A limit algebra is the inductive limit of a system of the form:

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} \cdots$$

where each A_i is a member of some class of finite dimensional algebras and each α_i is an injective algebra homomorphism, possibly satisfying additional specified properties. Limit algebras have become an important source for many varied examples of norm-closed non-self-adjoint algebras [B,HL,HPo,MS2,MS3,PePW1,PePW2,PeW,PW,Po2,Po4,SV,T,V1].

The most restrictive non-self-adjoint context is for each A_i to be the algebra of upper-triangular n_i by n_i matrices for some sequence n_i. A more general context is for each A_i to be the upper-triangular matrices in a finite-dimensional C*-algebra, i.e., a direct sum of full matrix algebras. The most general building blocks which have been fruitful to date are digraph algebras (finite-dimensional CSL algebras), as described in [Po2] or Section 6.7 of [Po4].

There is also a range of possible assumptions for the maps. We will assume that each map is unital, $*$-extendible and regular. A regular map is one that maps matrix units to sums of matrix units. This assumption ensures that there are enough partial isometries in $\mathcal{A}$ which normalize $\mathcal{A} \cap \mathcal{A}^*$ to span $\mathcal{A}$. For a discussion of limit algebras where the maps are not regular, see [Po3].

Using the $*$-extendibility of the maps, we can conclude that the limit algebra is contained, in the first context, in a UHF C*-algebra and in the second, in an AF C*-algebra. In fact, the C*-algebra generated by such a limit algebra is the C*-envelope of the limit algebra.

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Much of the literature has dealt with special families of limit algebras; these usually arise in one of two ways:

(1) all limit algebras possessing some natural intrinsic property,
(2) all limit algebras arising from direct systems with a particular class of embeddings.

This paper will deal primarily with the second situation. The focus is on embeddings which are order preserving, that is, which preserve the natural ordering on the diagonal (this is defined precisely in Section 2). In [Po4] and [Po5] such embeddings are called strongly regular but the term order preserving seems more natural to us.

Two issues immediately present themselves:

- Find intrinsic properties which characterize the family under consideration,
- Classify direct systems, i.e., when do two different presentations yield the same algebra (up to isometric isomorphism) ?

We will consider both of these questions. The first problem has been solved elsewhere for nest embeddings [HPe] and mixing embeddings [Do]; the second question has been answered for standard embeddings [B,PePW1,Po1], refinement embeddings [PePW1,Po1], and alternation embeddings [HPo,P]. The classification results in this paper subsume those for algebras with refinement, standard and alternation embeddings.

There are a number of other classification theorems in the literature. Algebras based on two special classes of nest embeddings, refinement with twist embeddings and homogeneous embeddings based on the backshift, were classified in [HPo]. In [Po3], Power classified an uncountable family of algebras based on nest embeddings which are not regular. The resulting nests generate masas which are not canonical. (Indeed, they generate singular masas.) Another classification theorem for regular, non-$*$-extendible embeddings between certain digraph algebras can be found in [Po2]. See also [Po7] for an additional example of a classification theorem.

In answering the first question, it is natural to consider elements of the limit algebra which preserve this diagonal order. These are also of interest in the study of product-type cocycles; see Lemmas 5.5 and 5.6 of [V1]. In the more general context of subalgebras of groupoid C*-algebras, the concept of a monotone G-set is equivalent to that of an order-preserving element; see page 57 of [MS1]. We thank Paul Muhly for pointing this out to us.

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Summary of Paper. In Section 1, we introduce the usual examples and notation for limit algebras; in Section 2, we define the order preserving normalizer, order preserving embeddings and locally order preserving embeddings. The main results of Section 2 are characterizations of locally order preserving embeddings between T_n's (Lemma 2), order preserving embeddings between T_n's (Theorem 5), and order preserving embeddings between direct sums of T_n's (Theorem 6).

In Section 3, we concentrate on the T_n context and characterize the spectra of such triangular AF algebras where the embeddings are locally order preserving (Theorem 7) and where compositions of the embeddings are locally order preserving (Theorem 8).

Section 4 describes in detail the spectrum of a TAF algebra with order preserving embeddings between T_n's. In particular, using arguments similar to those in [HPo] we obtain the first step in our classification of algebras which are limits of order preserving systems (Theorem 13). The spectral description is also useful in constructing explicit cocycles for these algebras, which shows these algebras and those with order preserving embeddings through direct sums of T_n's are analytic (Theorems 14 and 15).

In Section 5, we establish various equivalent conditions for a subset of the normalizer to generate the algebra (Proposition 17). Applying this to the order preserving normalizer, we show that for a triangular AF algebra, the order preserving normalizer generates the algebra if, and only if, it has a presentation $\varinjlim(A_i, \alpha_i)$ where $\alpha_j \circ \alpha_{j-1} \circ \cdots \circ \alpha_i$ is locally order preserving for each i and j.

In Section 6, we state and prove a theorem relating isometric isomorphisms and intertwining diagrams for inductive limits; this type of theorem has appeared implicitly in a paper of Davidson and Power, [DPo], and in slightly different forms as Theorem 4.6 of [V1] and Corollary 1.14 of [PeW]. We use this theorem to give simple proofs of several known classification theorems and to extend a recent result of Poon and Wagner [PW].

To apply the intertwining diagram theorem to classifying algebras with order preserving embeddings between T_n's, we need a unique factorization theorem for order preserving embeddings between T_n's. This is obtained in Section 7. Finally, in Section 8 we put together the results obtained in previous sections to give a classification theorem for algebras with order preserving embeddings between T_n's.

1 Preliminaries

Recall that a C*-algebra is *approximately finite* (AF) if there is a nested sequence of finite-dimensional C*-algebras whose closed union is the original C*-algebra. Given an AF C*-algebra $\mathcal{C}$ and a maximal abelian self-adjoint subalgebra (masa) $\mathcal{M} \subseteq \mathcal{C}$, we call $\mathcal{M}$ a *canonical masa* if there is a nested sequence of finite-dimensional C*-subalgebras of $\mathcal{C}$, $\{C_i\}$, so that

(1) $\mathcal{C} = \overline{\cup_i C_i}$,
(2) $M_i = C_i \cap \mathcal{M}$ is a masa in C_i for each i, and
(3) $N_{M_i}(C_i) \subseteq N_{M_{i+1}}(C_{i+1})$ for each i,

where

$$N_X(Y) = \{y \in Y \mid y \text{ is a partial isometry, and } y^*xy, yxy^* \in X \text{ for all } x \in X\}.$$

If D_n is the algebra of diagonal $n \times n$ matrices and T_n is the algebra of upper-triangular $n \times n$ matrices, then $N_{D_n}(T_n)$ consists of upper-triangular matrices with entries either 0 or of absolute value 1 such that each row or column has at most one non-zero entry.

We can now define a *triangular* AF (TAF) algebra, $\mathcal{A}$, to be a norm-closed subalgebra of an AF C*-algebra so that $\mathcal{A} \cap \mathcal{A}^*$ is a canonical masa in the AF C*-algebra. A *canonical algebra* is a norm-closed subalgebra of an AF C*-algebra that contains a canonical masa.

If we let A_i denote $\mathcal{A} \cap C_i$ and α_i denote the injection map from C_i to C_{i+1} restricted to A_i, then we have a system where each A_i is finite-dimensional and each α_i is $*$-extendible. We will call the system

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} \cdots \tag{1}$$

a *presentation* of the TAF algebra $\mathcal{A}$.

In general, each A_i is not maximal as a triangular subalgebra of C_i and need not be even if $\mathcal{A}$ is maximal as a triangular AF algebra in $\mathcal{C}$ (see [PePW1, Example 3.25] or [Po4, Proposition 10.4]). A TAF algebra is called *strongly maximal* if there is a sequence of finite-dimensional C*-algebras C_i as above so that A_i is maximal as a triangular subalgebra of C_i for each i. These are precisely the TAF algebras that have presentations such as (1) with each A_i either the upper-triangular $n \times n$ matrices for some n or a direct sum of such.

By an *embedding*, we mean an injective algebra homomorphism between triangular subalgebras of finite-dimensional C*-algebras that extends to an injective $*$-homomorphism of the C*-algebras and is *regular* in the sense that it maps matrix units to sums of matrix units. Algebras built from embeddings which are not $*$-extendible or from embeddings which are not regular have been studied in the literature [Po2,Po3,HL]; we are incorporating regularity and $*$-extendibility into our definition since all the embeddings that we study in this paper satisfy these two properties. We can thereby avoid endless repetition of these two assumptions. If $\alpha\colon A_1 \to A_2$ is a regular embedding, then $\alpha(N_{D_1}(A_1)) \subseteq N_{D_2}(A_2)$ where D_i is the diagonal $A_i \cap A_i^*$, for $i = 1, 2$. Since an embedding is $*$-extendible, if $\alpha\colon T_n \to T_m$ is an embedding, then n divides m; the *multiplicity* of α is the quotient m/n.

Two fundamental examples of embeddings have influenced much of the theory of TAF algebras. They are the *standard embedding*, $\sigma_k\colon T_n \to T_{nk}$, given by

$$\sigma_k(A) = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix},$$

where the righthand side is a k by k block matrix, and the *refinement embedding* (or the canonical nest embedding), $\rho_k\colon T_n \to T_{nk}$, given by

$$\rho_k((a_{ij})) = (a_{ij} I_k)\,,$$

where I_k is the k by k identity matrix.

2 Embeddings

Consider T_n and its diagonal projections, denoted by $\mathcal{P}(T_n)$. The diagonal ordering on $\mathcal{P}(T_n)$ (denoted $\preceq$) is a partial order given by

$$e \preceq f \iff \text{there exists } w \in N_{D_n}(T_n) \text{ with } ww^* = e,\ w^*w = f.$$

Notice that two comparable projections must have equal traces and that this ordering is a total order on the minimal diagonal projections.

Each element $w \in N_{D_n}(T_n)$ induces a partial homeomorphism on $\mathcal{P}(T_n)$, with domain $\{x \in \mathcal{P}(T_n) \,|\, x \leq ww^*\}$ and range $\{x \in \mathcal{P}(T_n) \,|\, x \leq w^*w\}$, given by $x \mapsto w^*xw$.

Definition. We say that w is *order preserving* if this map preserves the diagonal ordering restricted to its domain and range. Define

$$N^{op}_{D_n}(T_n) = \{w \in N_{D_n}(T_n) \,|\, w \text{ is order preserving}\}.$$

In other words, $w \in N^{op}_{D_n}(T_n)$ if, and only if, $x \prec y$ implies $w^*xw \prec w^*yw$ for *all* diagonal projections x, y with $x, y \leq ww^*$. Every matrix unit in T_n is trivially order preserving. The following partial isometry is not order preserving in T_4:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}.$$

Given $x, y \in N_{D_n}(T_n)$, $x + y \in N_{D_n}(T_n)$ if, and only if, the initial projections of x and y are orthogonal and the final projections are orthogonal. As the matrix above shows, the same is not true if we replace $N_{D_n}(T_n)$ with $N^{op}_{D_n}(T_n)$.

The definitions below make sense for embeddings which are regular; $*$-extendibility is not needed. All the order preserving embeddings in this paper will, however, be $*$-extendible.

Definition. An embedding ϕ is *locally order preserving* if $\phi(e)$ is order preserving for each matrix unit e. An embedding ϕ is *order preserving* if $\phi(N^{op}_{D_n}(T_n)) \subset N^{op}_{D_{nk}}(T_{nk})$.

The map $\phi\colon T_2 \to T_4$ given by

$$\phi\left(\begin{bmatrix} a & b \\ & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 & 0 & b \\ & a & b & 0 \\ & & c & 0 \\ & & & c \end{bmatrix}$$

is an example of an embedding which is *not* locally order preserving. On the other hand, the map $\psi\colon T_{2n} \to T_{4n}$ given by

$$\psi\left(\begin{bmatrix} A & B \\ & C \end{bmatrix}\right) = \begin{bmatrix} A & 0 & B & 0 \\ & A & 0 & B \\ & & C & 0 \\ & & & C \end{bmatrix}$$

with $A, C \in T_n$ and $B \in M_n$ is locally order preserving but not order preserving for $n > 1$. Since matrix units are in the order preserving normalizer, all order preserving maps are locally order preserving. Both refinement and standard embeddings are order preserving.

Much of the above discussion extends directly to direct sums of T_n's. Suppose $T = \oplus_{i=1}^{a} T_{m_i}$ and $D = T \cap T^*$. We can define a diagonal order, $\preceq$, for projections in T just as before. Notice that two minimal diagonal projections are comparable if, and only if, they are in the same summand.

Again, each $w \in N_D(T)$ induces a partial homeomorphism on $\mathcal{P}(T)$ given by $x \mapsto w^*xw$ and we can define $N_D^{op}(T)$ just as before. A key fact about $N_D^{op}(T)$ is that if $a, b \in N_D^{op}(T)$ are contained in *different* summands of T, then $a + b \in N_D^{op}(T)$.

Just as before, we can define locally order preserving and order preserving for an embedding $\phi\colon \bigoplus_{i=1}^{a} T_{m_i} \to \bigoplus_{j=1}^{b} T_{n_j}$.

In either context, it is easy to see that the composition of two order preserving embeddings is order preserving; the same is *not* true for locally order preserving embeddings (see, for instance, the first example on page 26 below). Nonetheless, we have the following lemma:

Lemma 1. *Let α and β be two embeddings.*
If $\beta \circ \alpha$ is locally order preserving, then α is locally order preserving.
If $\beta \circ \alpha$ is order preserving, then α is order preserving.

Proof. To prove the first statement, suppose α is not locally order preserving. Then there is a matrix unit e and there are minimal diagonal subprojections of $\alpha(ee^*)$, x and y, so that $x \prec y$ and $\alpha(e^*)x\alpha(e) \succ \alpha(e^*)y\alpha(e)$.

Since β is regular and $*$-extendible, it is easy to see that

$$\beta(x) \prec \beta(y) \text{ and } \beta(\alpha(e^*)x\alpha(e)) \succ \beta(\alpha(e^*)y\alpha(e)).$$

But, as β is a homomorphism, $\beta(\alpha(e^*)x\alpha(e)) = \beta(\alpha(e^*))\beta(x)\beta(\alpha(e))$. Thus, we have

$$\beta(x) \prec \beta(y) \text{ and } \beta(\alpha(e^*))\beta(x)\beta(\alpha(e)) \succ \beta(\alpha(e^*))\beta(y)\beta(\alpha(e)).$$

This shows the map $z \mapsto \beta(\alpha(e^*))z\beta(\alpha(e))$ is not order preserving on the two subprojections $\beta(x)$ and $\beta(y)$. Hence $\beta(\alpha(e))$ is not order preserving and the first statement is proved.

To prove the second statement, repeat the above argument with α not order preserving and e in the order preserving normalizer. $\square$

We turn now to characterizing locally order preserving embeddings and order preserving embeddings between T_n's.

Locally Order Preserving Embeddings. If $\phi: T_n \to T_{nk}$ is locally order preserving, then ϕ is determined by its action on D_n, and, in particular, by its action on the minimal diagonal projections of D_n.

It is helpful to let $[\,n\,] = \{1, 2, \dots, n\}$. Clearly, we can identify the minimal diagonal projections of D_n under the diagonal ordering with $[\,n\,]$ under the usual ordering. In the following, we use ordered pairs of integers as indices for minimal diagonal projections in D_{nk}. For the sake of clarity, we will denote such a projection by its index alone (i.e., we will write (i, j) for $e_{(i,j),(i,j)}$). There is a bijection between the minimal diagonal projections of D_{nk} and $[\,n\,] \times [\,k\,]$ as follows: For each $i \in [\,n\,]$, $\phi(e_{ii})$ is the sum of k minimal diagonal projections in D_{nk}; for $\phi(e_{11})$, index these projections in the order in which they appear in the diagonal order by $(1, 1), (1, 2), \dots, (1, k)$; for $\phi(e_{ii})$ in general, let (i, j) be the image of $(1, j)$ under conjugation by $\phi(e_{1i})$. With this indexing, we have

$$\phi(e_{i,j}) = \sum_{l=1}^{k} e_{(i,l),(j,l)}. \tag{2}$$

Notice that $\phi(e_{1j})$ is order preserving if, and only if, the diagonal ordering restricted to $\{(j, l) \mid l \in [k]\}$ induces the usual order on $[k]$. (An obvious identification is made here.) In general, $\phi(e_{ij})$ is order preserving if and only if, the diagonal ordering restricted to $\{(i, l) \mid l \in [k]\}$ and to $\{(j, l) \mid l \in [k]\}$ both induce the same order on $[k]$. In particular, if ϕ is locally order preserving, then the ordering on $[\,n\,] \times [\,k\,]$ satisfies:

$$\begin{aligned} i_1 < i_2 &\Longrightarrow (i_1, j) \preceq (i_2, j) \qquad \text{any } j \in [\,k\,] \\ j_1 < j_2 &\Longrightarrow (i, j_1) \preceq (i, j_2) \qquad \text{any } i \in [\,n\,] \end{aligned} \tag{3}$$

It is straightforward to prove that:

Lemma 2. *There is a bijection, given by (2), between locally order preserving embeddings $\phi: T_n \to T_{nk}$ and orderings on $[\,n\,] \times [\,k\,]$ that satisfy (3).*

This correspondence between local order preservation and the properties of the diagonal order on $[n] \times [k]$ depends on the $*$-extendibility of the embedding. For example, the compression embeddings considered in [HL] are outside this framework even though they satisfy the conditions in the definition of order preserving, since they are not $*$-extendible.

Order Preserving Embeddings. Given the correspondence in Lemma 2, it is natural to ask under what additional conditions does the diagonal ordering on $[\,n\,] \times [\,k\,]$ correspond to an order preserving map?

Lemma 3. *A locally order preserving embedding $\phi: T_n \to T_{nk}$ is order preserving if, and only if, there are no $a, b \in [\,k\,]$ so that*

$$(g, a) \prec (i, b) \text{ and } (h, a) \succ (j, b)$$

where g, h, i and j satisfy $e_{gh} + e_{ij} \in N^{op}_{D_n}(T_n)$.

One consequence of this lemma is that in checking if a map such as ϕ is order preserving, we need only consider those elements of the order preserving normalizer which are the sum of *two* matrix units.

Proof. To establish necessity, observe that conjugation by $\phi(e_{gh})$ maps (h, a) to (g, a) and conjugation by $\phi(e_{ij})$ maps (j, b) to (i, b). If there do exist $a, b \in [\,k\,]$ with the given properties, then conjugation by $\phi(e_{gh} + e_{ij})$ is not order preserving and so ϕ is not order preserving.

For sufficiency, it is enough to assume ϕ is not order preserving and find a, b, g, h, i, and j satisfying the conditions. By assumption, there is some $x \in N^{op}_{D_n}(T_n)$ so that $\phi(x) \notin N^{op}_{D_{n+1}}(T_{n+1})$. Thus there are two elements (h, a) and (j, b) so that conjugation by $\phi(x)$ reverses the diagonal ordering. Let (g, a) and (i, b) be their images under conjugation. It follows that this choice of a, b, g, h, i and j satisfies the conditions. $\square$

Lemma 4. *Let $\phi: T_n \to T_{nk}$ be an order preserving embedding. Let $t \in [k]$. The diagonal ordering on $[n] \times [k]$ satisfies*

$$(i, t) \prec (g, h) \prec (j, t) \Longrightarrow i \leq g \leq j$$

for all i and j.

Proof. Assume that $(i, t) \prec (g, h) \prec (j, t)$. If either $g < i$ or $g > j$ then

$$w = e_{gg} + e_{ij} \in N^{op}_{D_n}(T_n).$$

Observe that $\phi(w)$ carries (j, t) to (i, t) and (g, h) to (g, h). Since $(g, h) \prec (j, t)$ and $(g, h) \succ (i, t)$, $\phi(w)$ is not order preserving, a contradiction. So we must have $i \leq g \leq j$. $\square$

Given any two order preserving embeddings, $\alpha: T_n \to T_{na}$ and $\beta: T_n \to T_{nb}$, it is easy to check that $\alpha \oplus \beta: T_n \to T_{n(a+b)}$ is order preserving. Since every refinement embedding is order preserving, it follows that every direct sum of refinement embeddings is order preserving.

The family of embeddings which are direct sums of refinement embeddings includes all refinement embeddings (one summand only), all standard embeddings (each refinement embedding has multiplicity 1) and all embeddings of the form $\sigma \circ \rho$ (the refinement embeddings in the direct sum all have equal multiplicity). These are the embeddings which yield the alternation algebras studied in [HPo,P,Po6].

The next theorem shows that the direct sums of refinement embeddings are precisely the class of order preserving embeddings.

Theorem 5. *Suppose $\phi\colon T_n \to T_{nk}$ is an embedding. Then ϕ is order preserving if, and only if, ϕ is a direct sum of refinement embeddings.*

Remark. Theorem 5 provides a description of all order preserving embeddings in the context in which embeddings are regular and $*$-preserving – the context of this paper. Outside this setting, compression embeddings [HL] provide examples of order preserving embeddings which are not direct sums of refinements embeddings.

Proof. As we have remarked earlier, it is easy to see that a direct sum of refinement embeddings is order preserving. To prove the converse, consider the diagonal order induced on $[n] \times [k]$ by ϕ as in the discussion in the section on locally order preserving embeddings. The fact that the range of ϕ is contained in the upper triangular matrices implies that the first element in the order is (1,1). Let r_1 be the largest integer so that $(1,1), \dots, (1, r_1)$ are the first r_1 elements of the diagonal order. Lemma 4 implies that no (g,h) with $g \geq 3$ can appear in between $(1,t)$ and $(2,t)$, so $(2,1), \dots, (2, r_1)$ must follow in the diagonal order. We cannot have $(2, r_1+1)$ next, for then $(2, r_1+1)$ would precede $(1, r_1+1)$, an impossibility. We cannot have $(1, r_1+1)$ next, as Lemma 4 implies that between $(2, r_1)$ and $(3, r_1)$ we can only have elements (g,h) with $g = 2$ or $g = 3$. Arguing in the same way, we see that the next r_1 elements are $(3,1)$ through $(3, r_1)$ and so on until $(n,1)$ through (n, r_1).

Upper triangularity of the image (or the conditions for local order preservation) guarantee that the next element is $(1, r_1+1)$. Now let r_2 be such that the diagonal order runs $(1, r_1+1), \dots (1, r_1+r_2), (2, r_1+1)$. We may continue the argument as before until we finally obtain integers $r_1, \dots, r_t$ whose sum is k with the property that the embedding $\rho_{r_1} \oplus \cdots \oplus \rho_{r_t}$ induces the same diagonal order on $[n] \times [k]$ as ϕ does. Lemma 2 now yields the theorem. $\square$

More Order Preserving Embeddings. In extending Theorem 5 to characterize order preserving embeddings between direct sums of T_n's, we need the notion of an ordered Bratelli diagram, first described in [Po5]. These diagrams play a role in [HPS] and in [PW]; our definitions follow those of [PW].

Definition. Given non-empty finite sets V and W, an *ordered diagram* from V to W is a partially ordered set E and maps $r\colon E \to W$ and $s\colon E \to V$ such that $e, e' \in E$ are comparable if and only if $r(e) = r(e')$.

The sets V and W are the *vertices* of the diagram and E are the *edges*. We extend the definition slightly to describe order preserving maps between direct sums of T_n's.

Definition. Call (E, r, s, f) an *ordered diagram with multiplicity* if (E, r, s) is an ordered diagram as defined above, and f is a function from E to $\mathbb{N}$.

We call $f(e)$ the *multiplicity* of the edge e.

To an ordered diagram with multiplicity, we can associate a direct sum of refinement embeddings. Let (E, r, s, f) be an ordered diagram with multiplicity from $V = \{1, 2, \dots, a\}$

to $W = \{1, 2, \dots, b\}$. Given positive integers $m_1, m_2, \dots, m_a$, the ordered diagram with multiplicity determines a map $\phi\colon \oplus_{i=1}^a T_{m_i} \to \oplus_{j=1}^b T_{n_j}$ where $n_j = \sum_{r(e)=j} m_{s(e)} f(e)$ for each $j \in W$. The map ϕ is given by

$$\phi\left(\bigoplus_{i=1}^a t_i\right) = \bigoplus_{j=1}^b \left(\bigoplus_{r(e)=j} \rho_{f(e)}(t_{s(e)})\right).$$

It is important to stress that the inner direct sum in the definition of ϕ is ordered. That is, the diagonal ordering of the summands $\rho_{f(e)}(t_{s(e)})$ in T_{n_j} is given by the ordering of E restricted to $\{e \in E \mid r(e) = j\}$. This association is a slight generalization of the association between ordered diagrams and embeddings given in [Po5] and [PW].

The following definition formalizes the notion of when two ordered diagrams with multiplicity should be considered 'the same'.

Definition. Two ordered diagrams with multiplicity, say (E, r, s, f) and (E', r', s', f'), are *order equivalent* if there is an order preserving bijection $\Phi\colon E \to E'$ so that

$$r(e) = r'(\Phi(e)), s(e) = s'(\Phi(e)), \text{ and } f(e) = f'(\Phi(e)).$$

We write this as $(E, r, s, f) \cong^{ord} (E', r', s', f')$.

Given two order equivalent diagrams with multiplicity, they both induce the same embedding, providing we use the same algebras as domain for both embeddings.

Theorem 6. *An embedding $\phi\colon \oplus_{i=1}^a T_{m_i} \to \oplus_{j=1}^b T_{n_j}$ is order preserving if, and only if, there is an ordered diagram with multiplicity whose associated embedding is ϕ.*

Proof. One direction is obvious. For the other we must show that ϕ is a direct sum of embeddings, each of which is essentially a refinement embedding of one summand of the domain of ϕ. Furthermore, each direct summand of ϕ must be supported on a projection which is an interval from the lattice of invariant projections for the co-domain of ϕ. Also, it is clearly sufficient to prove the theorem for the case in which the co-domain of ϕ consists of a single full upper triangular matrix algebra.

Thus, we assume that $\phi\colon \oplus_{k=1}^a T_{m_k} \to T_n$ is order preserving. For each k, let $\{e_{ij}^k\}$ be a matrix unit system for T_{m_k} and let $\{f_{ij}\}$ be a matrix unit system for T_n. We also identify each e_{ij}^k with the obvious matrix unit in $\oplus_{k=1}^a T_{m_k}$. For clarity, let e_i^k denote the minimal diagonal projection e_{ii}^k and, similarly, let f_i denote f_{ii}. In each case, the usual order on the index set corresponds to the diagonal order on the minimal diagonal projections.

The following observation is critical: we cannot have three minimal diagonal projections, $f_b \prec f_c \prec f_d$ and unequal integers k and l such that f_c is subordinate to $\phi(e_p^k)$, for some p; f_b is subordinate to $\phi(e_n^l)$, for some n; f_d is subordinate to $\phi(e_m^l)$, for some m; and conjugation by $\phi(e_{nm}^l)$ carries f_d to f_b. The reason is that if $k \neq l$ then $e_{nm}^l + e_p^k$

is necessarily order preserving in $\oplus_{k=1}^{a} T_{m_k}$, but $\phi(e^l_{nm} + e^k_p) = \phi(e^l_{nm}) + \phi(e^k_p)$ is not. (Conjugation by the latter partial isometry maps f_c to f_c and f_d to f_b; but $f_c \prec f_d$ and $f_b \prec f_c$.)

Now consider f_1, the first diagonal projection in T_n. There is a unique index k such that f_1 is subordinate to $\phi(e^k_1)$. Let 1^k denote the projection $e^k_1 + \cdots + e^k_{m_k}$. This operator is the projection in the domain algebra for ϕ corresponding to the summand T_{m_k}. Let ψ denote the mapping obtained by restricting ϕ to T_{m_k} and also compressing to $\phi(1^k)$. Let s be the number of minimal diagonal projections f_i which are subordinate to $\phi(1^k)$. We retain the diagonal order on these projections inherited from T_n; with respect to this order, ψ is an order preserving embedding from T_{m_k} to T_s. By theorem 5, ψ is a direct sum of refinement embeddings.

Let ρ be the first summand of ψ (the one for which $\rho(e^k_1)$ contains f_1 as a subordinate). Let t be the multiplicity of ρ. Observe that each subordinate of $\rho(1^k)$ precedes all of the other subordinates of $\phi(1^k)$. This fact, combined with the critical observation above, implies that the subordinates of $\rho(1^k)$ are $f_1, f_2, \ldots, f_{tm_k}$. In other words, $\rho(1^k)$ is an interval from the nest associated with T_n (and a leading interval, at that).

It is now clear that we can split ρ off from ϕ as a direct summand and apply induction to what remains to see that ϕ must have the desired form. □

3 The Spectrum for Locally Order Preserving Embeddings

Having described embeddings that are locally order preserving, it is natural to consider the algebras $\varinjlim(T_{n_i}, \alpha_i)$ where each $\alpha_i \colon T_{n_i} \to T_{n_{i+1}}$ is locally order preserving. There is also a smaller class of algebras, properly contained in those with each α_i locally order preserving and properly containing those with each α_i order preserving. This class consists of all algebras $\varinjlim(A_i, \alpha_i)$ where for each i and j with $i < j$, we have $\alpha_{i,j}$ is locally order preserving, where $\alpha_{i,j}$ is the composition

$$\alpha_{j-1} \circ \alpha_{j-2} \circ \cdots \circ \alpha_{i+1} \circ \alpha_i.$$

We will later show (in Theorem 18) that this class is precisely all those strongly maximal TAF algebras where the order preserving normalizer generates the algebra. (The order preserving normalizer in a TAF algebra is defined in Section 5.)

The classification in Section 8 of limit algebras with order preserving presentations makes critical use of an invariant, sometimes called the *fundamental relation* but which we shall call the *spectrum*, for subalgebras of AF C*-algebras which contain a canonical masa. We introduce the latter term because this invariant plays a role analogous to the role played by the spectrum (i.e. the maximal ideal space) of an abelian C*-algebra. This invariant was first described in the form in which we need it in [Po1]. We shall describe it briefly; a more complete account may be found in [Po4].

Let $\mathcal{A}$ be a TAF algebra with diagonal $\mathcal{D}$ and let X be the maximal ideal space for $\mathcal{D}$. The spectrum for $\mathcal{A}$ is a topological binary relation, denoted by $R(\mathcal{A})$, on X. This

relation is determined by the normalizing partial isometries in $\mathcal{A}$. Since $\mathcal{D}$ is isomorphic to $C(X)$ and since each normalizing partial isometry acts by conjugation on $\mathcal{D}$, each normalizing partial isometry induces in a natural way a partial homeomorphism on X. (Partial, because the domain for the homeomorphism is the subset of X which corresponds to the initial projection of the partial isometry.) The spectrum is the union of the graphs of all these partial homeomorphisms; the topology is generated by taking each such graph as an open and closed subset of $R(\mathcal{A})$.

The spectrum can also be described in the language of groupoids. The enveloping C*-algebra for $\mathcal{A}$ is a groupoid C*-algebra; the spectrum for $\mathcal{A}$ is the support subsemigroupoid for the algebra $\mathcal{A}$. The main significance of the spectrum for us is that it is a complete isometric isomorphism invariant for triangular subalgebras of AF C*-algebras when the diagonal algebras are regular canonical masas. Effective use of the spectrum often requires calculating a specific representation for the spectrum. We do this in our context in this and the following section.

Locally Order Preserving Embeddings. Many of the spectra which have been described explicitly in the literature have a common form. Here we show this common form is precisely equivalent to the existence of a presentation with locally order preserving embeddings.

Consider a system with locally order preserving embeddings

$$T_{k_1} \xrightarrow{\phi_1} T_{k_1 k_2} \xrightarrow{\phi_2} T_{k_1 k_2 k_3} \xrightarrow{\phi_3} \dots \rightarrow \mathcal{A}$$

Let n_m denote $k_1 k_2 \cdot \ldots \cdot k_m$. By Lemma 2 we obtain, for each $m \in \mathbb{N}$:

A) A bijection between $[\,k_1\,] \times \cdots \times [\,k_m\,]$ and the minimal diagonal projections of D_{n_m} — we will define $X_m = [\,k_1\,] \times \cdots \times [\,k_m\,]$ as the index set for the minimal diagonal projections of D_{n_m}.

B) A total order on X_m (which we denote by $\preceq_m$) so that the bijection in A) is an order isomorphism. (The order on the minimal diagonal projections of D_{n_m} is the diagonal order.)

Furthermore, the indexing and order satisfy:

a) $\phi_m\colon T_{n_m} \to T_{n_{m+1}}$ is given on matrix units by the formula

$$\phi_m(e_{(x_1,\ldots,x_m),(y_1,\ldots,y_m)}) = \sum_{j=1}^{k_{m+1}} e_{(x_1,\ldots,x_m,j),(y_1,\ldots,y_m,j)}.$$

b) If $i < j$ then $(x_1, \ldots, x_m, i) \preceq_{m+1} (x_1, \ldots, x_m, j)$.

c) If $(x_1, \ldots, x_m) \preceq_m (y_1, \ldots, y_m)$ then $(x_1, \ldots, x_m, j) \preceq_{m+1} (y_1, \ldots, y_m, j)$.

Definition. A sequence of orders $\preceq_m$ on the sets X_m satisfying properties b) and c) is said to be *coherent*.

Let $X = \prod_{j=1}^{\infty} [\,k_j\,]$ and give X the product topology. Then X is isomorphic to the maximal ideal space of $\mathcal{D}$, the diagonal of $\mathcal{A}$, where $\mathcal{D} = \varinjlim D_{n_m}$. Let $R(\mathcal{A})$ denote the spectrum, a topological binary relation on X.

It is routine to show that the indexing above yields:

$$\begin{aligned} xR(\mathcal{A})y \iff &\text{There exists } m \in \mathbb{N} \text{ such that } x_n = y_n \text{ for all } n \geq m \\ &\text{and } (x_1, \ldots, x_m) \preceq_m (y_1, \ldots, y_m). \end{aligned}$$

While there are many choices for m, coherence guarantees that the initial segments are ordered the same way for any choice of m giving common tails.

More colloquially,

$$\begin{aligned} xR(\mathcal{A})y \iff &x \text{ and } y \text{ have the same tails and the initial segments} \\ &\text{are ordered with respect to a coherent sequence of orders.} \end{aligned}$$

Conversely, let $X = \prod_{j=1}^{\infty} [\,k_j\,]$, $X_m = \prod_{j=1}^{m} [\,k_j\,]$, and $\preceq_m$ be a total order on X_m. Assume that the sequence of orders is coherent.

Let R be a topological binary relation on X defined by:

$$\begin{aligned} xRy \iff & \text{There exists } m \in \mathbb{N} \text{ such that } x_n = y_n \text{ for all } n > m \\ & \text{and } (x_1, \dots, x_m) \preceq_m (y_1, \dots, y_m). \end{aligned}$$

The topology is given by taking (for each $m \in \mathbb{N}$ and $a, b \in X_m$) the following sets as a basis of clopen sets:

$$E_{a,b} = \{\, (x,y) \in X \times X : x_n = y_n \text{ for } n > m,\ x_n = a_n \text{ and } y_n = b_n \text{ for } 1 \le n \le m \,\}$$

It is clear that each $E_{a,b} \subset R$.

Definition. A topological binary relation is *coherent* if it is isomorphic (as a topological binary relation) to one of the form above. Actually, the form given above is only one representation of the topological binary relation, so it would be more precise to say that it is a topological binary relation *with a coherent representation.* For the sake of brevity, we use the shorter term.

Theorem 7. *If $\mathcal{A}$ is the direct limit of a system,*

$$T_{k_1} \xrightarrow{\phi_1} T_{k_1 k_2} \xrightarrow{\phi_2} T_{k_1 k_2 k_3} \xrightarrow{\phi_3} \dots \to \mathcal{A}$$

where ϕ_i is locally order preserving for each i, then $R(\mathcal{A})$ is coherent.

Conversely, if R is coherent, then $R \cong R(\mathcal{A})$ where $\mathcal{A}$ is a direct limit of such a system.

Proof. We have already proved the first statement in the theorem, so only the converse remains to be proved.

As usual, $n_m = k_1 \cdot \ldots \cdot k_m$ and $X_m = [\, k_1 \,] \times \cdots \times [\, k_m \,]$. Assume R is isomorphic to a relation with the form above. Use X_m as the index set for the minimal diagonal projections in D_{n_m}; order the minimal diagonal projections according to the order on X_m; let T_{n_m} be the algebra of upper triangular matrices with respect to this order. Define $\phi_m : D_{n_m} \to D_{n_{m+1}}$ by

$$\phi_m(e_{(x_1,\dots,x_m)}) = \sum_{j=1}^{k_{m+1}} e_{(x_1,\dots,x_m,j)}$$

and extend ϕ_m to a locally order preserving embedding $T_{n_m} \to T_{n_{m+1}}$. The coherence guarantees that $\phi_m(T_{n_m}) \subseteq T_{n_{m+1}}$.

Let $\mathcal{A} = \varinjlim(T_{n_m}, \phi_m)$. Then it is clear that $R \cong R(\mathcal{A})$. $\square$

An Intermediate Family. There is an analogue of Theorem 7 for the class of algebras $\varinjlim(A_i, \alpha_i)$ where each A_i is the upper triangular matrices of some full matrix algebra and each composition of embeddings $\alpha_{i,j} = \alpha_{j-1} \circ \cdots \circ \alpha_i$ is locally order preserving.

We may rephrase the second line of (3) in Section 2 as follows: the diagonal order restricted to $\{(i,l) \,|\, l \in [k]\}$ induces the same order on $[k]$, for each choice of $i \in [n]$. The appropriate generalization is that given i and j with $i < j$, then the diagonal order on X_j restricted to

$$\{(x_1, \dots, x_i, a_{i+1}, \dots, a_j) \,|\, a_l \in [k_l] \text{ for all } l \text{ with } i < l \leq j\}$$

induces the same order on $[k_{i+1}] \times [k_{i+2}] \times \cdots \times [k_j]$ for each choice of an element $(x_1, \dots, x_i)$ in X_i.

Definition. A spectrum with a coherent representation that satisfies this additional condition for all i and j with $i < j$ will be called *hypercoherent*.

Theorem 8. *If $\mathcal{A}$ is the direct limit of a system,*

$$T_{k_1} \xrightarrow{\phi_1} T_{k_1 k_2} \xrightarrow{\phi_2} T_{k_1 k_2 k_3} \xrightarrow{\phi_3} \dots \longrightarrow \mathcal{A}$$

where for all i and j with $i < j$, the composition $\phi_{i,j} = \phi_{j-1} \circ \cdots \circ \phi_i$ is locally order preserving, then $R(\mathcal{A})$ is hypercoherent.

Conversely, if R is hypercoherent, then $R \cong R(\mathcal{A})$ where $\mathcal{A}$ is a direct limit of such a system.

Proof. The following observation is the key to proving both directions: given a matrix unit $e = e_{(x_1,\dots,x_i),(y_1,\dots,y_i)} \in T_{k_1 \cdots k_i}$, then $\phi_{i,j}(e)$ is order preserving in $T_{k_1 \cdots k_j}$ if and only if the diagonal order on X_j restricted to

$$\{(x_1, \dots, x_i, a_{i+1}, \dots, a_j) \,|\, a_l \in [k_l] \text{ for all } l \text{ with } i < l \leq j\}$$

and

$$\{(y_1, \dots, y_i, a_{i+1}, \dots, a_j) \,|\, a_l \in [k_l] \text{ for all } l \text{ with } i < l \leq j\}$$

both induce the same order on $[k_{i+1}] \times [k_{i+2}] \times \cdots \times [k_j]$.

If $\mathcal{A}$ is the direct limit of a system with each $\phi_{i,j}$ locally order preserving, then the observation immediately implies that $R(\mathcal{A})$ is hypercoherent. Conversely, given a hypercoherent spectrum R, by Theorem 7 we can construct a system

$$T_{k_1} \xrightarrow{\phi_1} T_{k_1 k_2} \xrightarrow{\phi_2} T_{k_1 k_2 k_3} \xrightarrow{\phi_3} \dots \longrightarrow \mathcal{A}$$

with locally order preserving embeddings such that $R \cong R(\mathcal{A})$. As $R(\mathcal{A})$ is hypercoherent, the observation implies that for each i and j with $i < j$, and for each matrix unit $e \in T_{k_1 \cdots k_i}$, the image $\phi_{i,j}(e)$ is order preserving in $T_{k_1 \cdots k_j}$. Thus, each $\phi_{i,j}$ is locally order preserving, as required. $\square$

4 Spectra for Order Preserving Embeddings

We turn now to the spectrum for algebras $\varinjlim(T_{n_i}, \alpha_i)$ where each $\alpha_i \colon T_{n_i} \to T_{n_{i+1}}$ is order preserving. After describing these spectra in the first subsection, we characterize the gap points in Theorem 9 and then, in Theorem 13, prove the first step in our classification of such algebras. The last two theorems of the section show that these algebras are analytic as are the algebras $\varinjlim(A_i, \alpha_i)$, where the A_i are allowed to be direct sums of T_n's and the α_i are still order preserving.

The Spectrum. Let, for each n, $\phi_{r^{(n)}} = \rho_{r_1^{(n)}} \oplus \cdots \oplus \rho_{r_{t_n}^{(n)}}$ where $r^{(n)}$ is the t_n-tuple $(r_1^{(n)}, \dots, r_{t_n}^{(n)})$ and consider the direct system:

$$T_1 \xrightarrow{\phi_{r^{(1)}}} T_{k_1} \xrightarrow{\phi_{r^{(2)}}} T_{k_1 k_2} \xrightarrow{\phi_{r^{(3)}}} \dots \longrightarrow \mathcal{A}.$$

Since direct sums of refinement embeddings are order preserving, we can apply the results of the previous section. In this case, we describe explicitly the hypercoherent orderings which extend naturally the lexicographic orders used for refinement embeddings and the reverse lexicographic orders used for standard embeddings. This description also subsumes that given in [HPo] for the spectrum of an alternation algebra.

To fix notation, note that the maximal ideal space of $\mathcal{A}$ is isomorphic to $X = \prod_{n=1}^{\infty} [\, k_n \,]$ where $k_n = r_1^{(n)} + \cdots + r_{t_n}^{(n)}$. Let $F_1^{(n)}, \dots, F_{t_n}^{(n)}$ be the partition of $[\, k_n \,]$ corresponding to $r_1^{(n)}, \dots, r_{t_n}^{(n)}$. Specifically,

$$\begin{aligned} F_1^{(n)} &= \{\, 1, \dots, r_1^{(n)} \,\}, \\ F_2^{(n)} &= \{\, r_1^{(n)} + 1, \dots, r_1^{(n)} + r_2^{(n)} \,\}, \\ &\vdots \\ F_{t_n}^{(n)} &= \{\, r_1^{(n)} + \cdots + r_{t_n - 1}^{(n)} + 1, \dots, k_n \,\}. \end{aligned}$$

For each integer $x \in [\, k_n \,]$, define $i_n(x)$ to be the unique index s so that $x \in F_s^{(n)}$.

Applying Lemma 2 to $\phi_{r^{(1)}} \colon T_{k_1} \to T_{k_1 k_2}$ gives the following ordering on $[\, k_1 \,] \times [\, k_2 \,]$:

$$(x_1, x_2) \preceq_2 (y_1, y_2) \qquad \text{if} \begin{cases} i_2(x_2) < i_2(y_2), \text{ or} \\ i_2(x_2) = i_2(y_2) \text{ and } x_1 < y_1, \text{ or} \\ i_2(x_2) = i_2(y_2), \ x_1 = y_1, \text{ and } x_2 \le y_2. \end{cases}$$

Next, define a total order on each of the sets $[\, k_1 \,] \times \cdots \times [\, k_n \,]$ recursively:

(1) The order $\preceq_2$ is defined on $[\, k_1 \,] \times [\, k_2 \,]$ as above.

(2) The order $\preceq_n$ is defined on $[\, k_1\,] \times \cdots \times [\, k_n\,]$ by treating it as the Cartesian product of $[\, k_1\,] \times \cdots \times [\, k_{n-1}\,]$ carrying the order $\preceq_{n-1}$ and $[\, k_n\,]$ with the usual total order and again applying the procedure above.

By Theorem 7, $xR(\mathcal{A})y$ if, and only if, there exists $m \in \mathbb{N}$ such that $x_n = y_n$ for all $n > m$ and $(x_1, \dots, x_m) \preceq_m (y_1, \dots, y_m)$. In terms of the i_n functions, if $xR(\mathcal{A})y$, then

(1) $x = y$, or
(2) There is a q such that $i_q(x_q) < i_q(y_q)$ and $i_n(x_n) = i_n(y_n)$ for all $n > q$, or,
(3) $i_n(x_n) = i_n(y_n)$ for all n, and there is a q such that $x_q < y_q$ and $x_m = y_m$ for all $m < q$.

Informally, to determine whether $xR(\mathcal{A})y$ or $yR(\mathcal{A})x$ for x and y with the same tails we compare the initial segments of x and y. First we look for the highest coordinates which belong in different $F^{(n)}$-sets; if this does not occur, we look for the lowest coordinates which differ. The order of the $F^{(n)}$-sets or of the coordinates themselves, as appropriate, determines the order between x and y. In short, the order is reverse lexicographical for $F^{(n)}$-sets, then lexicographical for the coordinates themselves.

Gap Points. The material in this section on gap points and the next on first refinement multiplicities follows the line of argument in [HPo], there given for the special case of alternation algebras.

The description of the gap points is easy; the verification that the description is correct is tedious.

Let $\mathcal{O}(x) = \{\, z \mid zR(\mathcal{A})x \,\}$ be the orbit of x and $\overline{\mathcal{O}(x)}$ the closure of the orbit.

Definition. We define x to be a *gap point* if there is a point y such that $y \notin \overline{\mathcal{O}(x)}$ and $\overline{\mathcal{O}(y)} = \overline{\mathcal{O}(x)} \cup \{\, y\,\}$.

Remark. It would be more accurate to call x a *left gap point*. Then y is the corresponding *right gap point* and could well be denoted by x^+.

There is one possible exception to the characterization of gap points in the following theorem. Let x^∞ denote the sequence (k_n). Thus, each coordinate of x^∞ is the maximal element of the $F^{(n)}$-set with maximal index. This point is exceptional for the condition below only in the case of a refinement algebra.

Theorem 9. *A point $x \neq x^\infty$ is a gap point if, and only if, there is an integer p so that for all $n > p$, $i_n(x_n) = \max F_1^{(n)}$. In other words, for large n, x_n is the largest element of the first $F^{(n)}$-set; viz., $x_n = r_1^{(n)}$.*

As we shall see, the gap points effectively determine up to a finite factor the supernatural number of the sequence $(r_1^{(n)})$ of multiplicities of the first refinement summands.

We shall need several preliminary facts about orbits in order to prove the theorem. For a point $y \in X$, define

$$W_p(y) = \{\, z \in X \mid z_1 = y_1, \dots, z_p = y_p \,\}.$$

When y is clearly understood, we write W_p instead. These sets are open, as well as closed, and form a basis for the topology at y. Convergence in the topology is pointwise:

$$z(n) \to z \iff z(n)_i \to z_i \text{ as } n \to \infty, \text{ for each } i.$$

Lemma 10 below, together with the observation that if x is a gap point, then $\mathcal{O}(x)$ is not dense in X, will establish the necessity of the condition $x_n \in F_1^{(n)}$ for all large n.

Lemma 10. *Assume that there are infinitely many n with $x_n \in F_p^{(n)}$ for $p > 1$. Then $\mathcal{O}(x)$ is dense in X.*

Proof. Let $y \in X$ be arbitrary. We need merely show that, for each $p > 0$, W_p contains a point in the orbit $\mathcal{O}(x)$. Given p, choose $n > p$ such that $x_n \notin F_1^{(n)}$. Let z_n be any element in $F_1^{(n)}$. For $1 \le t \le p$, let $z_t = y_t$. For all other $t \ne n$, let $z_t = x_t$. Then $z = (z_1, z_2, \dots)$ lies in $W_p \cap \mathcal{O}(x)$. Thus, $\mathcal{O}(x)$ is dense in X. □

Let

$$\begin{aligned} A_p &= \{\, x \in X \mid x_t \le r_1^{(t)} \text{ for } t \ge p \,\} \\ &= \{\, x \in X \mid i_t(x_t) = 1 \text{ for } t \ge p \,\} \\ &= \prod_{n=1}^{p-1} [\, k_n \,] \times \prod_{n=p}^{\infty} F_1^{(n)} \end{aligned}$$

Observe that each A_p is a closed set.

Now suppose that $y \in X$ and $\mathcal{O}(y)$ is not dense in X. Then there exists an integer p such that $y_n \in F_1^{(n)}$, for all $n \ge p$. If $x \in \mathcal{O}(y)$, then $i_n(x_n) = 1$ for all $n \ge p$ also, so $\mathcal{O}(y) \subseteq A_p$. Since A_p is closed, $\overline{\mathcal{O}(y)} \subseteq A_p$.

With y as above, suppose that $x \in A_p$ and that $(x_1, \dots, x_q) \preceq (y_1, \dots, y_q)$ for some $q \ge p$. Then $(x_1, \dots, x_n) \preceq (y_1, \dots, y_n)$ for all n satisfying $p \le n \le q$. This follows immediately from the definition of $\preceq$ and the fact that $i_n(x_n) = 1 = i_n(y_n)$ for $p \le n \le q$. If we actually have $(x_1, \dots, x_q) \prec (y_1, \dots, y_q)$, then we also have $(x_1, \dots, x_n) \prec (y_1, \dots, y_n)$ for all $n \ge q$.

With y still as above and x in $\mathcal{O}(y)$, there is an integer $q \ge p$ such that $y_n = x_n$ for all $n \ge q$. If $y \ne x$, then $(x_1, \dots, x_q) \prec (y_1, \dots, y_q)$. Consequently,

$$x \in \mathcal{O}(y) \Longrightarrow (x_1, \dots, x_n) \preceq (y_1, \dots, y_n) \text{ for all } n \ge p.$$

Lemma 11. *Assume that $\mathcal{O}(y)$ is not dense in X, i.e. that there is an integer p such that $i_n(y_n) = 1$, for all $n \ge p$. Then*

$$x \in \overline{\mathcal{O}(y)} \iff \textit{there exists } s \ge p \textit{ such that } (x_1, \dots, x_n) \preceq (y_1, \dots, y_n), \textit{ for all } n \ge s.$$

Proof. Let $x \in \overline{\mathcal{O}(y)}$. Let $n \geq p$ and let W_n be the corresponding neighborhood of x. There is an element $z \in \mathcal{O}(y)$ such that $z \in W_n$. Consequently, $(x_1, \dots, x_n) = (z_1, \dots, z_n) \preceq (y_1, \dots, y_n)$. This establishes the implication $\Rightarrow$.

For the converse, we may assume, based on the remarks above, that $(x_1, \dots, x_n) \preceq (y_1, \dots, y_n)$ for all $n \geq p$. Fix $n \geq p$ and define a point $z \in X$ by

$$z_t = \begin{cases} y_t & \text{if } 1 \leq t \leq n, \\ x_t & \text{if } n < t. \end{cases}$$

Then $z \in \mathcal{O}(y)$ and $z \in W_n$. Thus, every neighborhood of x intersects $\mathcal{O}(y)$, hence $x \in \overline{\mathcal{O}(y)}$. □

Corollary 12. *If $x \in \overline{\mathcal{O}(y)}$ then $\overline{\mathcal{O}(x)} \subseteq \overline{\mathcal{O}(y)}$.*

Proof. Apply Lemma 11. □

We are now ready to prove Theorem 9.

Proof of Theorem 9. Let x be a (left) gap point. Lemma 10 shows that there is an integer p such that $i_n(x_n) = 1$ for all $n \geq p$. We need to show further that $x_n = \max F_1^{(n)}$ for all but finitely many n.

Assume the contrary; that is, assume that $i_n(x_n) = 1$ for all $n \geq p$ but that $x_n \neq \max F_1^{(n)}$ for infinitely many values of n. We must show that there is no point y which satisfies $y \notin \overline{\mathcal{O}(x)}$ and $\overline{\mathcal{O}(y)} = \overline{\mathcal{O}(x)} \cup \{y\}$. Clearly, y cannot satisfy these two properties if $\mathcal{O}(y)$ is dense in X or if $\overline{\mathcal{O}(x)} \nsubseteq \overline{\mathcal{O}(y)}$. So we may reduce to the case in which y satisfies: $x \in \overline{\mathcal{O}(y)}$, $x \neq y$ and there is an integer q such that $i_n(y_n) = 1$ for all $n \geq q$. Without loss of generality, we may assume that $q \geq p$.

By Lemma 11 there is an integer $s \geq q$ such that

$$(x_1, \dots, x_n) \prec (y_1, \dots, y_n) \text{ for all } n \geq s.$$

Let m be any integer such that $m > s$ and $x_m \neq \max F_1^{(m)}$. Define $z = z(m)$ by

$$z_t = \begin{cases} x_t & \text{if } t \neq m, \\ x_m + 1 & \text{if } t = m. \end{cases}$$

It is evident from Lemma 11 that $z \notin \overline{\mathcal{O}(x)}$. Lemma 11 also shows that $z \in \overline{\mathcal{O}(y)}$. Indeed, $i_t(z_t) = i_t(x_t) = 1 = i_t(y_t)$ for all $t > s$ and $(z_1, \dots, z_s) \prec (y_1, \dots, y_s)$. (Note: this uses $i_m(z_m) = i_m(x_m + 1) = 1$.) Consequently, $(z_1, \dots, z_n) \prec (y_1, \dots, y_n)$ for all $n \geq s$ and $z \in \overline{\mathcal{O}(y)}$, as desired.

Since there are infinitely many m such that $x_m \neq \max F_1^{(m)}$, there are infinitely many distinct points $z(m)$ which are in $\overline{\mathcal{O}(y)}$ but not in $\overline{\mathcal{O}(x)}$. Thus, in this case, $\overline{\mathcal{O}(y)} \neq \overline{\mathcal{O}(x)} \cup \{y\}$.

Next, we prove the converse. So assume that x satisfies $x_n = \max F_1^{(n)}$ for all $n \geq p$. We must produce an element $y \in X$ such that $y \notin \overline{\mathcal{O}(x)}$ and $\overline{\mathcal{O}(y)} = \overline{\mathcal{O}(x)} \cup \{y\}$. We consider separately two cases.

Case 1. There is an integer m such that $x_m \neq \max F_j^{(m)}$ and $x_n = \max F_j^{(n)}$ for $n > m$. (Here, $j = i_m(x_m)$ or $j = i_n(x_n)$ as appropriate. Of course, if $n \geq p$ then $j = 1$. Also, note that $x_t \neq \max F_j^{(t)}$ is possible for only finitely many t, so there must be a maximal such t if there are any.)

Define $y \in X$ by:

$$y_t = \begin{cases} x_t, & \text{if } 1 \leq t \leq m-1, \\ x_m + 1, & \text{if } t = m, \\ \min F_j^{(t)}, & \text{if } t > m, \quad j = i_t(x_t). \end{cases}$$

We have $i_n(y_n) = i_n(x_n)$ for all n, and in particular, $i_n(y_n) = 1$ for all $n \geq p$. From this and the fact that $x_m < y_m$, it is clear that $(x_1, \dots, x_n) \prec (y_1, \dots, y_n)$ for all $n \geq m$. So $x \in \overline{\mathcal{O}(y)}$ and hence (by Corollary 12) $\overline{\mathcal{O}(x)} \subseteq \overline{\mathcal{O}(y)}$. It is also clear that $y \notin \overline{\mathcal{O}(x)}$.

It remains to show that if $z \in \overline{\mathcal{O}(y)}$ and $z \neq y$, then $z \in \overline{\mathcal{O}(x)}$. Given such a z, let $r > m$ be such that $(z_1, \dots, z_n) \prec (y_1, \dots, y_n)$ for all $n \geq r$. It suffices to show that $(z_1, \dots, z_n) \preceq (x_1, \dots, x_n)$ for all $n \geq r$. Fix $n \geq r$. If $i_t(z_t) < i_t(y_t)$ for some $t \leq n$ (a possibility only if $t < p$), then $(z_1, \dots, z_n) \prec (x_1, \dots, x_n)$. If $i_t(z_t) = i_t(y_t)$ for all t, then there is an integer q such that $z_q < y_q$ and $z_t = y_t$ for all $t < q$. We cannot have $q > m$, since $y_t = \min F_j^{(t)}$ for $t > m$. If $q < m$, the $(z_1, \dots, z_n) \prec (x_1, \dots, x_n)$ as desired. If $q = m$ then $z_m \leq x_m$. If $z_m < x_m$, we again have $(z_1, \dots, z_n) \prec (x_1, \dots, x_n)$. Finally, if $z_m = x_m$ then, since $i_t(x_t) = \max F_j^{(t)}$ for all $t > m$ and $i_t(z_t) = i_t(y_t) = i_t(x_t)$ for all t, we have $(z_1, \dots, z_n) \preceq (x_1, \dots, x_n)$. This exhausts all possibilities and case 1 is complete.

Case 2. For all n, $x_n = \max F_j^{(n)}$, where $j = i_n(x_n)$. Let q be the least integer such that $x_q \neq k_q$, i.e., $i_q(x_q)$ is not maximal in the set of indices for $F^{(q)}$. There must be such an integer q, since we assume that x is not the exceptional point x^∞. (This could, in fact, happen only if the algebra is actually a refinement algebra: $F_1^{(n)} = [\, k_n \,]$ for all large n.) We now define y by:

$$y_t = \begin{cases} 1, & \text{if } 1 \leq t \leq q-1, \\ x_q + 1, & \text{if } t = q, \\ \min F_j^{(t)}, & \text{if } t > q \text{ and } j = i_t(x_t). \end{cases}$$

Observe that $i_t(y_t) = 1$ if $1 \leq t \leq q-1$, that $i_q(y_q) = i_q(x_q) + 1$, and that $i_t(y_t) = i_t(x_t)$ for all $t > q$. In particular, $y_t = \min F_j^{(t)}$ for all t.

For $n \geq q$ it is clear that $(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$. This implies that $y \notin \overline{\mathcal{O}(x)}$ and $x \in \overline{\mathcal{O}(y)}$, which, by Corollary 12, implies that $\overline{\mathcal{O}(x)} \subset \overline{\mathcal{O}(y)}$. It remains to prove that $\overline{\mathcal{O}(y)} = \overline{\mathcal{O}(x)} \cup \{y\}$.

Let $z \in \overline{\mathcal{O}(y)}$ be such that $z \neq y$. We must have $i_t(z_t) = 1 = i_t(y_t)$ for all large t. We claim that $i_t(z_t) \neq i_t(y_t)$ for some t. Indeed, if $i_t(z_t) = i_t(y_t)$ for all t, then the two facts $(z_1, \ldots, z_n) \preceq (y_1, \ldots, y_n)$ for all large n and $y_t = \min F_j^{(t)}$ for all t imply that $(z_1, \ldots, z_n) = (y_1, \ldots, y_n)$ for all large n. But this means that $z = y$.

Thus, there is an integer s such that $i_s(z_s) < i_s(y_s)$ and $i_t(z_t) = i_t(y_t)$ for all $t > s$. Note that $s < \max\{\,p, q\,\}$, since if $t \geq \max\{\,p, q\,\}$, then $i_t(y_t) = i_t(x_t) = 1$. If $s > q$ then $i_s(z_s) < i_s(y_s) = i_s(x_s)$ and $i_t(z_t) = i_t(y_t) = i_t(x_t)$ for all $t > s$. Hence, $(z_1, \ldots, z_n) \prec (x_1, \ldots, x_n)$ for all $n \geq s$ and $z \in \overline{\mathcal{O}(x)}$. We cannot have $s < q$, since if we do, $i_s(y_s) = 1$, which is incompatible with $i_s(z_s) < i_s(y_s)$. So we are left with the case in which $s = q$. We then have $i_s(z_s) \leq i_s(x_s)$. For $t < s$, $i_t(x_t)$ is maximal, so it follows that $i_t(z_t) \leq i_t(x_t)$ for all t. Since $x_t = \max F_j^{(t)}$ for all t, we have $(z_1, \ldots, z_n) \preceq (x_1, \ldots, x_n)$ for all n.

Thus in all cases, $z \in \overline{\mathcal{O}(x)}$. This completes the proof of the theorem. $\square$

First Summand Refinement Multiplicities. Let x be a gap point and let $y \in \mathcal{O}(x)$. Let p be an integer such that $i_n(x_n) = 1$ for all $n \geq p$ and $x_n = \max F_1^{(n)}$ for all $n \geq p$. We may assume that p is the least integer with these properties, though it is not actually necessary to do so. Lemma 11 implies

$$\overline{\mathcal{O}(x)} = \{\, z \colon (z_1, \ldots, z_{p-1}) \preceq (x_1, \ldots, x_{p-1}) \text{ and } z_n \in F_1^{(n)} \text{ for } n \geq p\,\}.$$

If we let h denote the number of elements of $[\,k_1\,] \times \cdots \times [\,k_{p-1}\,]$ which precede $(x_1, \ldots, x_{p-1})$, then we have

$$\overline{\mathcal{O}(x)} \cong [\,h\,] \times \prod_{n=p}^{\infty} [\,r_1^{(n)}\,].$$

We endow $\overline{\mathcal{O}(x)}$ with the relative topology and order that it inherits from $R(\mathcal{A})$ and $[\,h\,] \times \prod_{n=p}^{\infty} [\,r_1^{(n)}\,]$ with the usual spectrum associated with a refinement algebra with supernatural number $h \prod_{n=p}^{\infty} r_1^{(n)}$.

Theorem 13. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two direct limits of systems where the embeddings are order preserving and hence are direct sums of refinement embeddings. Assume that $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic. Let $(r_1^{(n)})$ be the sequence of multiplicities of the first refinement summand in the embeddings for $\mathcal{A}$. Let $(s_1^{(n)})$ be the corresponding sequence for $\mathcal{B}$. Let $sn(r)$ and $sn(s)$ be the supernatural numbers for these two sequences. Then there are finite numbers a and b such that $a\, sn(r) = b\, sn(s)$.*

Proof. Let x be the point with $x_n = \max F_1^{(n)}$ for all n. We assume that $\mathcal{A}$ and $\mathcal{B}$ are not refinement algebras, since a stronger result is known in that case $(\mathrm{sn}(r) = \mathrm{sn}(s))$. So x

is not an exceptional point. Let β be a spectrum isomorphism of $R(\mathcal{A})$ onto $R(\mathcal{B})$. Then $\beta(x)$ is a gap point in $R(\mathcal{B})$ and β restricted to $\overline{\mathcal{O}(x)}$ is a spectrum isomorphism of $\overline{\mathcal{O}(x)}$ onto $\overline{\mathcal{O}(\beta(x))}$. The description of closed orbits above yields the theorem. $\square$

Analyticity. The detailed description of the spectrum for a limit algebra with order preserving embeddings makes it fairly simple to prove that these algebras are all analytic.

Analytic algebras have been studied in detail in papers such as [V1,PePW2,SV,PW]; we refer the reader to these sources for a complete description of the notion. A practical working definition of analyticity can be given in terms of the existence of a cocycle on the spectrum of the algebra. If R is a topological binary relation, a *cocycle* c on R is a continuous function $c\colon R \longrightarrow \mathbb{R}$ satisfying the "cocycle" property: $c(x,y)+c(y,z) = c(x,z)$ for all x, y and z such that $(x,y) \in R$ and $(y,z) \in R$. (Technically, c is a 1-cocycle.) In our situation, where R is the spectrum of an AF algebra, R will be a relation on a compact Hausdorff topological space X (the maximal ideal space of a canonical masa). We sometimes find it convenient, albeit a little imprecise, to refer to c as a cocycle on X.

A cocycle is *locally constant* if it is constant on some neighborhood of each point of its domain. Locally constant cocycles have been studied in [VW,V2]. Clearly, a locally constant function is always continuous.

Theorem 14. *If $\mathcal{A} = \varinjlim(T_{n_i}, \alpha_i)$ is a direct limit with every α_i order preserving, then $\mathcal{A}$ is an analytic algebra. Furthermore, there is a locally constant cocycle defined on the spectrum of $\mathcal{A}$.*

Proof. We may assume that the spectrum, $R(\mathcal{A})$, of $\mathcal{A}$ is a topological binary relation defined on the set $X = \prod_{i=1}^{\infty}[\,k_i\,]$ in the fashion described at the beginning of this section. Let $X_m = \prod_{i=1}^{m}[\,k_i\,]$, for each positive integer m. The sequence of sets X_m carries a coherent family of orders $\preceq_m$ which determines the order on X. In order to define a cocycle c on X, it will suffice to define a sequence of cocycles c_m on X_m which satisfy the properties:

(1) $c_m(x,y) \geq 0$ if, and only if $x \preceq_m y$, for all $x, y \in X_m$.
(2) If $x = (x_1, \dots, x_m) \in X_m$ and $j \in [\,k_{m+1}\,]$, let (x,j) denote $(x_1, \dots, x_m, j)$, an element of X_{m+1}. Then $c_{m+1}((x,j),(y,j)) = c_m(x,y)$, for all $x, y \in X_m$ and $j \in [\,k_{m+1}\,]$.

Indeed, given such a sequence of cocycles, we can define a cocycle c on X as follows: if $xR(\mathcal{A})y$ then there is an integer p such that $x_i = y_i$ for all $i \geq p$. Define $c(x,y) = c_p((x_1, \dots, x_p),(y_1, \dots, y_p))$. Property (2) guarantees that c is well-defined. The construction makes it clear that c is locally constant, and hence continuous. Property (1) ensures that $\mathcal{A}$ is the analytic algebra determined by c.

Since $X_1 = [\,k_1\,]$ and $\preceq_1$ is the usual order on integers, c_1 is uniquely determined by specifying the $k_1 - 1$ values $c(i, i+1)$, for $1 \leq i \leq k_1 - 1$. The only constraint imposed by the properties above is that these numbers all be positive. After c_1 has been selected,

we can define the remaining c_m's recursively. The recursive step is identical at each stage; furthermore, a change in notation will make this step much easier to write down.

Since X_{m-1} is totally ordered by $\preceq_{m-1}$, this set is order isomorphic to $[\,k\,]$ with the usual order, where k is the cardinality of X_{m-1}. So in place of c_{m-1}, we may assume that we have a cocycle c_1 defined on $[\,k\,]$ with the property that $c(i,j) \geq 0$ whenever $i \leq j$. For simplicity of notation, let n denote k_m and $\preceq$ denote $\preceq_m$. Let $Y = [\,k\,] \times [\,n\,]$. If $r_1, \dots, r_p$ are the refinement multiplicities of the embedding associated with this step, then $n = r_1 + \cdots + r_p$ and the order on Y can be described as follows: the first elements of Y are the elements of $[\,k\,] \times F_1 = [\,k\,] \times [\,r_1\,]$ in the lexicographic order. Next come all the elements of $[\,k\,] \times F_2$, again in the lexicographic order. Continue with the groups $[\,k\,] \times F_3, \dots, [\,k\,] \times F_p$, always with the lexicographic order within each group. The sets F_i are the sets defined on page 16.

Our task then is to define a cocycle c_2 on Y subject to the properties:

(1) $c_2(x,y) \geq 0$ if, and only if $x \preceq y$.
(2) $c_2((i,t),(j,t)) = c_1(i,j)$ for all $i,j \in [\,k\,]$ and all $t \in [\,n\,]$.

We will define c_2 at all pairs (α, β) where β is the immediate successor of α in Y under the order $\preceq$. The cocycle property then determines c_2. If α is the last element in the group $[\,k\,] \times F_t$ and β is the first element in the next group, then $c_2(\alpha,\beta)$ may be chosen arbitrarily, so long as it is positive. (This is the case in which $\alpha = (k, r_1 + \cdots + r_t)$ and $\beta = (1, r_1 + \cdots + r_t + 1)$.)

Now suppose that $\alpha \in [\,k\,] \times F_t$ and that α is not the last element of this subset of Y. There are two cases to be distinguished. In the first, $\alpha = (i,j)$, where $i \in [\,k\,]$ and where j satisfies $r_1 + \cdots + r_{t-1} + 1 \leq j < r_1 + \cdots + r_t$. In this case, the immediate successor to α is $\beta = (i, j+1)$. If we let $d = \min\{c_1(i,i+1) \,|\, 1 \leq i \leq k-1\}$, then we may define $c_2(\alpha,\beta) = \dfrac{d}{r_t}$. In the second case, $j = r_1 + \cdots + r_t$ and $i < k$; the immediate successor of α is $\beta = (i+1, r_1 + \dots r_{t-1} + 1)$. In this situation, we define $c_2(\alpha,\beta) = c_1(i,i+1) - \dfrac{r_t - 1}{r_t} d$.

We have now defined c_2 at all pairs (α,β), where β is the immediate successor of α. There is a unique extension of c_2 to a cocycle defined on all of Y and this extension satisfies the two required properties. □

Remark. The proof in Theorem 14 can easily be extended to a slightly more general situation: the case in which the "building block" algebras are maximal triangular subalgebras of finite dimensional C*-algebras; i.e., direct sums of T_n's. The proof of the theorem proceeds by defining the locally constant cocycle on subsets of the spectrum which are graphs of matrix units. This is thinly disguised by the reductions which were made in order to achieve notational simplification. The specific representation of the spectrum is convenient for expressing the proof but is not essential; use of the sets $[\,k_1\,] \times \cdots \times [\,k_n\,]$ as index sets for the matrix units in corresponding finite dimensional algebras would enable one to define the cocycle on the graphs of the matrix units in the spectrum.

In the argument presented in the theorem, the construction of the cocycle c_2 on the image of each refinement embedding (more precisely, on the graphs of the matrix units in each image) does not depend on using the same cocycle c_1 on the domain for each refinement embedding. Consequently, if we had an order preserving embedding from a direct sum of T_n's into a single T_n with a different cocycle associated with each summand of the domain, we could still use the same procedure to construct a cocycle for the range. If the range is also a direct sum of T_n's, we simply construct a cocycle for each summand of the range and put these together. Thus, the following is true:

Theorem 15. *If $\mathcal{A}$ is a triangular subalgebra of an AF C^*-algebra and if $\mathcal{A}$ has a presentation with order preserving embeddings between direct sums of T_n's, then $\mathcal{A}$ is an analytic algebra with a locally constant cocycle defined on its spectrum.*

5 Intrinsic Characterizations

In this section we use Proposition 17 to obtain an intrinsic characterization result, Theorem 18; however, the Proposition itself is perhaps of some interest. Let $\mathcal{PI}(\mathcal{D})$ denote the partial isometries in $\mathcal{D}$. We need the following lemma, which is equivalent to Lemma 3.5 (c) of [PePW1]; for completeness, we give a proof.

Lemma 16. *Let $\mathcal{A}$ be a canonical algebra containing a canonical masa $\mathcal{D}$. If x and y are elements of $N_{\mathcal{D}}(\mathcal{A})$ with $\|x-y\|<1$, then $x^*y \in \mathcal{PI}(\mathcal{D})$.*

Proof. First, we show x and y have the same initial and final projections. Suppose x^*x and y^*y are different. Then there is a projection p that is a subprojection of one and orthogonal to the other. Without loss of generality, assume $p \le x^*x$ and $p \perp y^*y$. Then $xp = (x-y)p$ so $1 = \|xp\| = \|(x-y)p\| < 1$. This proves that x and y have the same initial projections; similarly, x and y have the same final projections.

Clearly, x^*y is a partial isometry. To prove $x^*y \in \mathcal{D}$, it suffices to prove that x^*y commutes with all projections in $\mathcal{D}$, or equivalently, with all subprojections of $xx^* = yy^*$. Let p be such a projection. We claim that $xpx^* = ypy^*$. Accepting this for the moment, we have

$$(x^*y)p = (x^*y)p(y^*y) = x^*(ypy^*)y = x^*(xpx^*)y = (x^*x)p(x^*y) = p(x^*y),$$

as required.

To prove the claim, suppose $xpx^* \neq ypy^*$. Then there is some subprojection of one that is orthogonal to the other, say q. Without loss of generality, assume $q \le xpx^*$ and $q \perp ypy^*$. Then $qxp \neq 0$ and $qyp = 0$, so $qxp = q(x-y)p$. It follows that $1 = \|qxp\| = \|q(x-y)p\| < 1$; a contradiction that proves the claim. $\square$

In the following Proposition, $R(\mathcal{A})$ denotes the spectrum for $\mathcal{A}$ defined on the maximal ideal space of the canonical masa $\mathcal{D}$ and, for each $e \in N_{\mathcal{D}}(\mathcal{A})$, $G(e)$ denotes the graph of the partial homeomorphism induced by e on the maximal ideal space of $\mathcal{D}$. Note that $G(e)$ is a compact, open subspace of $R(\mathcal{A})$.

Proposition 17. *Let $\mathcal{A}$ be a canonical algebra containing a canonical masa $\mathcal{D}$. Suppose $X \subset N_{\mathcal{D}}(\mathcal{A})$ satisfies $\mathcal{PI}(\mathcal{D}) \cdot X \subseteq X$ (or equivalently $X \cdot \mathcal{PI}(\mathcal{D}) \subseteq X$). Then the following are equivalent:*

(1) *The closed span of X is $\mathcal{A}$.*
(2) *Each element of $N_{\mathcal{D}}(\mathcal{A})$ can be written as a finite sum of elements of X.*
(3) *In any presentation for $\mathcal{A}$, each matrix unit can be written as a finite sum of matrix units in X.*
(4) $R(\mathcal{A}) = \bigcup_{x \in X} G(x)$.

If in addition $X \cdot X \subset X$, then we have another equivalent condition:

(5) *There is a presentation of $\mathcal{A}$, $\varinjlim(B_i, \phi_i)$, with each matrix unit in each B_i in X.*

Proof. $(1 \Rightarrow 2)$ Let $y \in N_{\mathcal{D}}(\mathcal{A})$. By hypothesis, the closed span of X is $\mathcal{A}$; so, there is some x, a finite linear combination of elements of X, such that $\|y - x\| < 1/2$. Write x as $\sum_{i=1}^{l} a_i x_i$, where a_i is a scalar and $x_i \in X$. By rewriting the sum and restricting x to the initial projection of y (which we can do as $X \cdot \mathcal{P}(\mathcal{D}) \subseteq X$), we may assume that each $x_i x_i^*$ is a subprojection of yy^* orthogonal to all other $x_j x_j^*$'s.

If $\sum_{i=1}^{l} x_i x_i^*$ does not sum to yy^*, then letting $z = yy^* - \sum_{i=1}^{l} x_i x_i^*$ we have $z(y-x) = zy$ and $\|z\| = 1$. This gives a contradiction, since then $\|y - x\| \geq \|z(y - x)\| = \|zy\| = 1$. Thus $yy^* = \sum_{i=1}^{l} x_i x_i^*$, where each $x_i \in X$. It follows that

$$y = yy^*y = \left(\sum_{i=1}^{n} x_i x_i^*\right) y = \sum_{i=1}^{n} x_i (x_i^* y).$$

Since the initial and final projections of each x_i are pairwise orthogonal, $\|x - y\| < 1/2$ implies $|1 - a_i| < 1/2$. Letting $x' = \sum_{i=1}^{l} x_i$, it follows that $\|x' - y\| = \|x' - x\| + \|x - y\| < 1$. As $x_i = x_i x_i^* x'$, we have

$$\|x_i - x_i x_i^* y\| \leq \|x_i x_i^*\| \|x' - y\| = \|x' - y\| < 1$$

and so by Lemma 16, $x_i^* x_i x_i^* y = x_i^* y \in \mathcal{PI}(\mathcal{D})$. Since $x_i \in X$ and $x_i^* y \in \mathcal{PI}(\mathcal{D})$, $x_i(x_i^* y) \in X$. Thus y is a finite sum of elements in X, as claimed.

$(2 \Rightarrow 3)$ Let $\varinjlim(A_i, \tau_i)$ be a presentation of $\mathcal{A}$ and let t be a matrix unit in A_a for some a. By (2), $t = x_1 + x_2 + \cdots + x_n$ where each $x_n \in X$. By Lemma 5.5 of [Po4], every element of $N_{\mathcal{D}}(\mathcal{A})$ (and in particular, each x_n) can be written as a partial isometry in $\mathcal{D}$ times a finite sum of matrix units. There is some A_b, $b \geq a$, that contains the finitely many matrix units in the sums for $x_1, \ldots, x_n$. Then we have $t = y_1 + y_2 + \cdots + y_m$ where each y_j is a partial isometry in $\mathcal{D}$ times a matrix unit in A_b. Since each y_j is part of a sum that gives some $x_i \in X$, by multiplying x_i on the left by the projection $y_j y_j^*$ we have each $y_j \in X$.

We may assume that no sum of y_a's equals zero by deleting all y_a's in such a sum. Since t also equals a sum of matrix units in A_b, say $z_1 + \cdots + z_l$, we have that

$$y_1 + \cdots + y_m - z_1 - z_2 - \cdots - z_l = 0.$$

This is only possible if $m = l$ and $y_1, \ldots, y_m$ is a permutation of $z_1, \ldots, z_l$. Thus t is a finite sum of matrix units in X. (Remark: here is another place where we use the assumption that embeddings are regular.)

$(3 \Rightarrow 1)$ This is immediate, as the closed span of $N_{\mathcal{D}}(\mathcal{A})$ is $\mathcal{A}$ and (3) implies the closed span of X contains $N_{\mathcal{D}}(\mathcal{A})$.

$(2 \Rightarrow 4)$ Obvious from the definition of $R(\mathcal{A})$.

$(4 \Rightarrow 2)$ Let $e \in N_{\mathcal{D}}(\mathcal{A})$. Since $G(e) \subset R(\mathcal{A}) = \bigcup_{x \in X} G(x)$ and $G(e)$ is compact, there exist $x_1, \ldots, x_n$ in X such that $G(e) \subseteq \bigcup_{i=1}^n G(x_i)$. By multiplying the x_i by suitable projections from $\mathcal{D}$, we may assume that $G(e) = \bigcup_{i=1}^n G(x_i)$ and that the $G(x_i)$ are pairwise disjoint. It now follows that $e = \sum_{i=1}^n p_i x_i$ for some $p_i \in \mathcal{PI}(\mathcal{D})$. By hypothesis, each $p_i x_i \in X$ and condition (2) holds.

We now also assume that $X \cdot X \subseteq X$ and prove that (5) is equivalent to the first four conditions.

$(3 \Rightarrow 5)$ Again, let $\varinjlim(A_i, \tau_i)$ be a presentation of $\mathcal{A}$. For each i, let X_i be the set of all matrix units in $A_i \cap X$. Let B_i be the closed span of X_i for each i. Observe that if $x, y \in X_i$ and $xy \neq 0$, then $xy \in X_i$ so B_i is a subalgebra of A_i.

We claim that $\tau_i(B_i) \subset B_{i+1}$. It suffices to show that if $x \in X_i$ and y is a matrix unit in A_{i+1} that appears in the sum of matrix units $\tau_i(x)$, then $y \in X_{i+1}$. However, $x \in X_i$ implies $\tau_i(x) \in X$. Since X is closed under multiplication by projections in the diagonal, the matrix unit $y = (yy^*)\tau_i(x)$ is in X and so in X_{i+1}.

The hypothesis is that, for each j, each matrix unit in A_j can be written as a finite sum of elements in X; hence $A_j \subset \cup_i B_i$. This implies that $\overline{\cup_i B_i} = \overline{\cup_i A_i} = \mathcal{A}$. The presentation $\mathcal{A} = \varinjlim(B_i, \tau_i|_{B_i})$ now has the required properties.

$(5 \Rightarrow 1)$ We can repeat the (short) argument given in $(3 \Rightarrow 1)$. $\square$

Proposition 17 yields an intrinsic characterization of those TAF algebra which have a presentation $\varinjlim(A_i, \phi_i)$ where each $\phi_{i,j} = \phi_j \circ \cdots \circ \phi_{i+1} \circ \phi_i$ is locally order preserving.

First, we need a few definitions. Let $\mathcal{A}$ be a subalgebra of an AF C*-algebra $\mathcal{B}$ containing a canonical masa $\mathcal{D}$. Following [PePW1], the diagonal ordering on the projections in $\mathcal{D}$, $\mathcal{P}(\mathcal{D})$, is given by

$$p \prec q \iff \text{there exists } w \in N_{\mathcal{D}}(\mathcal{A}) \text{ with } ww^* = p \text{ and } w^*w = q.$$

Given $w \in N_{\mathcal{D}}(\mathcal{A})$, there is a partial homeomorphism given by $x \mapsto w^*xw$ with domain $\{x \in \mathcal{P}(\mathcal{D}) \,|\, x \leq ww^*\}$ and range $\{x \in \mathcal{P}(\mathcal{D}) \,|\, x \leq w^*w\}$. Call w *order preserving* if this

map preserves the diagonal ordering restricted to its domain and range. Let

$$N_{\mathcal{D}}^{op}(\mathcal{A}) = \{w \in N_{\mathcal{D}}(\mathcal{A}) \,|\, w \text{ is order preserving}\}.$$

If $\mathcal{A} = T_n$, this agrees with the definition of $N_{D_n}^{op}(T_n)$ in Section 2. Also note that $N_{\mathcal{D}}^{op}(\mathcal{A})$ is intrinsic; it does not depend on choosing a presentation for $\mathcal{A}$.

We should remark that $e \in N_{\mathcal{D}}^{op}(\mathcal{A})$ if and only $G(e) \times G(e)$, as a partial homeomorphism on $X \times X$, sends $R(\mathcal{A})$ into $R(\mathcal{A})$. (As usual, X is the maximal ideal space of $\mathcal{D}$.) For subalgebras of groupoid C*-algebras, such subsets of the spectrum (support subsemigroupoid) are called *monotone G-sets* (page 57 of [MS1]). Groupoids admitting a cover of monotone G-sets (i.e., those satisfying condition (3) below) have arisen in the study of prime ideals; see Theorem 4.5 of [MS1].

Theorem 18. *Let $\mathcal{A}$ be a canonical algebra containing a canonical masa $\mathcal{D}$. The following are equivalent:*

1. *The closed span of $N_{\mathcal{D}}^{op}(\mathcal{A})$ is $\mathcal{A}$.*
2. *$\mathcal{A}$ has a presentation $\varinjlim(A_i, \phi_i)$ so that, for all i and j, each $\phi_{i,j} = \phi_{j-1} \circ \cdots \circ \phi_i$ is locally order preserving.*
3. *$R(\mathcal{A}) = \bigcup\{G(e) \,|\, e \in N_{\mathcal{D}}^{op}(\mathcal{A})\}$.*

Proof. Since $N_{\mathcal{D}}^{op}(\mathcal{A})$ satisfies all the conditions on X in Proposition 17, if the closed span of $N_{\mathcal{D}}^{op}(\mathcal{A})$ is $\mathcal{A}$ then $\mathcal{A}$ has a presentation $\varinjlim(A_i, \phi_i)$ with each matrix unit in $N_{\mathcal{D}}^{op}(\mathcal{A})$. It follows that each $\phi_{i,j}$ is locally order preserving.

The other direction follows from the observation that if all the $\phi_{i,j}$ are locally order preserving, then every matrix unit is in $N_{\mathcal{D}}^{op}(\mathcal{A})$.

The equivalence of the condition that the spectrum is the union of the graphs of the normalizing partial isometries with the other two conditions follows immediately from Proposition 17. □

Remark. If $\mathcal{A}$ is a strongly maximal TAF algebra, then we can choose the presentation in condition (2) to be strongly maximal, i.e., each A_i maximal triangular in C*(A_i). To see this, notice that in $(3 \Rightarrow 5)$ of the proof of Proposition 17, if A_i is maximal triangular in C*(A_i) then B_i is maximal triangular in C*(B_i). Thus in Proposition 17, if $\mathcal{A}$ is strongly maximal then it follows that the presentation in condition (5) can be chosen to be strongly maximal and similarly in Theorem 18.

The next two examples show that the class of algebras in Theorem 18 is properly contained in the class of algebras with locally order preserving presentations and properly contains the class of all algebras with order preserving presentations using direct sums of T_n's. The first example appears in [Do] as Example 13. A similar example can be found in [PePW2] (Example 3.7).

Example. Since locally order preserving embeddings are determined by their action on the diagonal, we can define a locally order preserving embedding $\phi_n: T_{3^n} \to T_{3^{n+1}}$ by specifying the values of ϕ_n on the minimal diagonal projections in T_{3^n}:

$$\begin{aligned}
\phi_n(e_1^{(n)}) &= e_1^{(n+1)} + e_2^{(n+1)} + e_4^{(n+1)}, \\
\phi_n(e_i^{(n)}) &= e_{3i-3}^{(n+1)} + e_{3i-1}^{(n+1)} + e_{3i+1}^{(n+1)}, \qquad \text{for } 1 < i < 3^n, \\
\phi_n(e_{3^n}^{(n)}) &= e_{3^{n+1}-3}^{(n+1)} + e_{3^{n+1}-1}^{(n+1)} + e_{3^{n+1}}^{(n+1)}.
\end{aligned}$$

Routine calculations will show that the composition of two successive embeddings in this system fails to be locally order preserving. This alone is not sufficient to show that the limit algebra obtained from this system fails to satisfy the conditions of Theorem 18, since there could, in principle, be other presentations which satisfy property (2).

A matrix unit $e_{ij}^{(n)}$ in T_{3^n} can be identified with its image in the limit algebra, $\mathcal{A}$. Observe that $e_{ij}^{(n)} \in N_{\mathcal{D}}^{op}(\mathcal{A})$ if, and only if, $\phi_{m,n}(e_{ij}^{(n)})$ is order preserving for all $m > n$. Using this, it is not hard to determine which matrix units in T_{3^n} are in the order preserving normalizer of $\mathcal{A}$. Indeed, it turns out that for $j > 1$, $e_{1j}^{(n)}$ is not in $N_{\mathcal{D}}^{op}(\mathcal{A})$ and for $i < 3^n$, $e_{i3^n}^{(n)}$ is not in $N_{\mathcal{D}}^{op}(\mathcal{A})$. All other matrix units are in the order preserving normalizer.

From these observations, it is clear that the order preserving normalizer fails to span $\mathcal{A}$, so the algebra $\mathcal{A}$ is not in the family characterized in Theorem 18. It is also illuminating to note that $\bigcup\{G(e) \mid e \in N_{\mathcal{D}}^{op}(\mathcal{A})\}$ is an open, dense, proper subset of $R(\mathcal{A})$.

The next example shows that $\overline{\operatorname{span} N_{\mathcal{D}}^{op}(\mathcal{A})} = \mathcal{A}$ is not sufficient to imply the existence of a presentation with order preserving embeddings between direct sums of T_n's. It has appeared before in the literature, as Example 3.2 in [SV], where it is shown to be a strongly maximal TAF algebra that is not analytic.

Example. Define $\phi_n: T_{2^n} \to T_{2^{n+1}}$ by

$$\begin{bmatrix} A & B \\ & C \end{bmatrix} \mapsto \begin{bmatrix} A & & B & \\ & A & & B \\ & & C & \\ & & & C \end{bmatrix}$$

where $A, C \in T_{2^{n-1}}$ and $B \in M_{2^{n-1}}$. While ϕ_n is not order preserving (consider $e_{1,1} + e_{2,1+2^{n-1}}$), it does map an order preserving sum of matrix units in A to an order preserving sum. Similar statements are true for order preserving sums in B or C.

If $\mathcal{A} = \varinjlim(T_{2^n}, \phi_n)$, then it is elementary to see that each matrix unit is in $N_{\mathcal{D}}^{op}(\mathcal{A})$ and so the closed span of $N_{\mathcal{D}}^{op}(\mathcal{A})$ is $\mathcal{A}$. As we noted above, [SV] shows that this algebra is not analytic and hence by Theorem 15, it cannot have a presentation using order preserving embeddings through direct sums of T_n's.

6 Intertwining Diagrams

Recall that a Banach algebra $\mathcal{A}$ is the *inductive limit* of the system

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} \cdots \tag{4}$$

if there exists a sequence of injective homomorphisms $\rho_i\colon A_i \to \mathcal{A}$ so that $\mathcal{A} = \overline{\cup_i \rho_i(A_i)}$ and the diagram

$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[dr, "\rho_i"'] & A_{i+1} \arrow[d, "\rho_{i+1}"] \\ & \mathcal{A} \end{tikzcd}$$

commutes for every i. Note that the injections ρ_i and subalgebras $\rho_i(A_i)$ are *not* unique in general. For example, if α is an automorphism of $\mathcal{A}$, then each ρ_i can be replaced by $\alpha \circ \rho_i$. We will use this freedom in proving Theorem 19, below.

Also, note that if $\{n_i\}$ is any increasing sequence of positive integers and $\alpha_{x,y} = \alpha_{y-1} \circ \cdots \circ \alpha_{x+1} \circ \alpha_x$ then both (4) and

$$A_{n_1} \xrightarrow{\alpha_{n_1,n_2}} A_{n_2} \xrightarrow{\alpha_{n_2,n_3}} A_{n_3} \xrightarrow{\alpha_{n_3,n_4}} A_{n_4} \xrightarrow{\alpha_{n_4,n_5}} \cdots$$

have the same inductive limit.

The following theorem appears elsewhere in the literature (Theorem 4.6 in [V1], Corollary 1.14 in [PeW] and the proof of Proposition 4.1 in [DPo]) in a somewhat different form. We include a proof for the sake of completeness; it is similar to the proof in [V1] but does not use the spectrum or cocycles. Also, the proof has a straightforward extension to bimodules over canonical masas. Note that Theorem 19 is not applicable without the assumptions of regularity and $*$-extendibility for the embeddings in the direct limits.

Theorem 19. *Let $\mathcal{A} = \varinjlim(A_i, \alpha_i)$ and $\mathcal{B} = \varinjlim(B_i, \beta_i)$ be norm-closed subalgebras of AF* C**-algebras containing canonical masas $\mathcal{C}$ and $\mathcal{D}$ respectively.*

If $\Phi\colon \mathcal{A} \to \mathcal{B}$ is an isometric algebra isomorphism with $\Phi(\mathcal{C}) = \mathcal{D}$, then there exist strictly increasing sequences of integers, $\{m_i\}$ and $\{n_i\}$ and regular, isometric homomorphisms ϕ_i so that the diagram

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_{1,m_1}"] \arrow[dr, "\phi_1"] & A_{m_1} \arrow[r, "\alpha_{m_1,m_2}"] \arrow[dr, "\phi_3"] & A_{m_2} \arrow[r, "\alpha_{m_2,m_3}"] \arrow[dr, "\phi_5"] & A_{m_3} \arrow[r, "\alpha_{m_3,m_4}"] \arrow[dr, "\phi_7"] & A_{m_4} \arrow[r, "\alpha_{m_4,m_5}"] \arrow[dr, "\phi_9"] & \cdots \mathcal{A} \arrow[d, leftrightarrow, "\Phi{,}\Phi^{-1}"] \\ B_1 \arrow[r, "\beta_{1,n_1}"] & B_{n_1} \arrow[r, "\beta_{n_1,n_2}"] \arrow[u, "\phi_2"] & B_{n_2} \arrow[r, "\beta_{n_2,n_3}"] \arrow[u, "\phi_4"] & B_{n_3} \arrow[r, "\beta_{n_3,n_4}"] \arrow[u, "\phi_6"] & B_{n_4} \arrow[r, "\beta_{n_4,n_5}"] \arrow[u, "\phi_8"] & \cdots \mathcal{B} \end{tikzcd} \tag{5}$$

commutes. Moreover, however we identify each A_i with an isomorphic subalgebra of $\mathcal{A}$ we can identify each B_i with an isomorphic subalgebra of $\mathcal{B}$ so that $\phi_{2i+1} = \Phi|_{A_{m_i}}$ and $\phi_{2i} = \Phi^{-1}|_{B_{n_i}}$.

Given a diagram such as (5) with each ϕ_i isometric, the universal property of inductive limits allows one to construct an isometric isomorphism between the inductive limits. What

is useful to us is the conclusion that every isometric isomorphism between TAF algebras arises in this way.

Also, notice that if $\mathcal{A}$ and $\mathcal{B}$ are triangular, then any isometric isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ necessarily satisfies $\Phi(\mathcal{A} \cap \mathcal{A}^*) = \mathcal{B} \cap \mathcal{B}^*$ and so the assumption that $\Phi(\mathcal{C}) = \mathcal{D}$ is automatically satisfied.

Proof. By the definition of inductive limit, there are nested subalgebras of $\mathcal{A}$ and $\mathcal{B}$ isomorphic to the algebras A_i and B_i, respectively. It is convenient to identify each A_i and B_i with its isomorphic subalgebra.

Let E be the set of matrix units in $\mathcal{A}$ given by this identification and similarly let G be the set of matrix units in $\mathcal{B}$. Let $F = \Phi(E)$, a second set of matrix units in $\mathcal{B}$.

Since $\Phi(\mathcal{C}) = \mathcal{D}$, it follows from Proposition 7.2 in [Po4] that $\Phi(N_{\mathcal{C}}(\mathcal{A})) = N_{\mathcal{D}}(\mathcal{B})$. For each $g \in G$, Lemma 5.5 in [Po4] implies that g is a partial isometry in $\mathcal{D}$ times a sum of elements of F. Suppose that $g = \delta_g f_g$ with δ_g a partial isometry in $\mathcal{D}$ and f_g a sum of elements in F. Without loss of generality, we may assume $gg^* = \delta_g \delta_g^* = \delta_g^* \delta_g = f_g f_g^*$ and $g^*g = f_g^* f_g$.

Define a map $\Gamma\colon G \to \mathcal{B}$ by $\Gamma(g) = f_g$. Notice that if $g = \sum_{i=1}^n g_i$ and $g, g_1, \dots, g_n \in G$, then $\delta_g f_g = \sum_{i=1}^n \delta_{g_i} f_{g_i}$. As $g \in G$, the matrix units g_i are pairwise orthogonal and hence so are the matrix units f_{g_i} and the partial isometries δ_{g_i}. Thus

$$\delta_g f_g = \sum_{i=1}^n \delta_{g_i} f_{g_i} = \sum_{i=1}^n \delta_{g_i} \sum_{i=1}^n f_{g_i}.$$

Multiplying by δ_g^* on the left and $\sum_{i=1}^n f_{g_i}^*$ on the right, we have

$$\delta_g^* \delta_g f_g \sum_{i=1}^n f_{g_i}^* = \delta_g^* \left(\sum_{i=1}^n \delta_{g_i} \right) \left(\sum_{i=1}^n f_{g_i} \right) \sum_{i=1}^n f_{g_i}^*.$$

As $\delta_g^* \delta_g f_g = f_g f_g^* f_g = f_g$ and similarly $\sum_{i=1}^n \delta_{g_i} \sum_{i=1}^n f_{g_i} \sum_{i=1}^n f_{g_i}^* = \sum_{i=1}^n \delta_{g_i}$, it follows that $f_g \sum_{i=1}^n f_{g_i}^* \in \mathcal{D}$. Since f_g and $\sum_{i=1}^n f_{g_i}$ are sums of matrix units and have the same initial and final projections, $f_g \neq \sum_{i=1}^n f_{g_i}$ would contradict $f_g \sum_{i=1}^n f_{g_i}^* \in \mathcal{D}$. Thus $f_g = \sum_{i=1}^n f_{g_i}$ or equivalently $\Gamma(g) = \sum_{i=1}^n \Gamma(g_i)$. It follows that we can extend Γ by linearity to $\cup_i B_i$ in a well-defined way.

Claim: $\Gamma\colon \cup_i B_i \to \cup_i \Phi(A_i)$ is an isometric automorphism and a $\mathcal{D}$-bimodule map. Accepting this for the moment, it follows that Γ can be extended to an isometric automorphism of $\mathcal{B}$. Replacing each B_i with the isomorphic subalgebra $\Gamma(B_i)$ does not change the presentation of $\mathcal{B}$. However, by the definition of Γ, any matrix unit in $F = \Phi(E)$ is a sum of matrix units in $\Gamma(G)$. Conversely, the image under Φ^{-1} of any matrix unit in $\Gamma(G)$ is a sum of matrix units in E.

So with respect to the systems of matrix units E and $\Gamma(G)$, Φ and Φ^{-1} map matrix units to sums of matrix units. Since there are only finitely many such matrix units in

A_1 and each is mapped to a finite sum of matrix units in $\mathcal{B}$, there is some n_1 so that $\Phi(A_1) \subset B_{n_1}$. Continuing in this way, we have the required sequences and can obtain each ϕ_i as the restriction of Φ or Φ^{-1}.

It remains only to prove the claim. By construction, Γ is linear. Suppose $g = g_1 g_2$ where $g_1, g_2 \in G$. Then $\delta_g f_g = \delta_{g_1} f_{g_1} \delta_{g_2} f_{g_2}$, so

$$f_g = (\delta_g^* \delta_{g_1}) f_{g_1} f_{g_1}^* f_{g_1} \delta_{g_2} f_{g_2} = (\delta_g^* \delta_{g_1}) f_{g_1} \delta_{g_2} f_{g_1}^* f_{g_1} f_{g_2} = (\delta_g^* \delta_{g_1})(f_{g_1} \delta_{g_2} f_{g_1}^*) f_{g_1} f_{g_2}.$$

Hence $f_g f_{g_2}^* f_{g_1}^* \in \mathcal{D}$. Again f_g and $f_{g_1} f_{g_2}$ are sums of matrix units with the same initial and final projections so arguing as before, we have $\Gamma(g_1 g_2) = \Gamma(g_1)\Gamma(g_2)$. Since every element of $\cup_i B_i$ is a linear combination of elements of G, it follows that Γ is multiplicative.

Observe that if $g \in G \cap \mathcal{D}$, then $g = f_g$ and $\Gamma(g) = g$. To see this, we first observe that $g \in \mathcal{D}$ and $f_g = \delta_g^* g$ so $f_g \in \mathcal{D}$. As g and f_g are projections, so is δ_g and hence $f_g = f_g f_g^* = \delta_g^* \delta_g = \delta_g$. Thus, $g = \delta_g f_g = f_g f_g = f_g$. It follows that $\Gamma(d_1 b d_2) = d_1 \Gamma(b) d_2$ for $d_1, d_2 \in \mathcal{D}$ and $b \in \cup_i B_i$, so Γ is a $\mathcal{D}$-bimodule map.

If f is a matrix unit in $\cup_i \Phi(A_i)$, then it can be written as the product of a partial isometry in $\mathcal{D}$ and a sum of matrix units in G, say $f = \epsilon \sum_i g_i$. On the other hand, each $g_i = \delta_{g_i} f_{g_i}$. Since Γ is a $\mathcal{D}$-bimodule map,

$$\Gamma\left(\epsilon \sum_i \delta_{g_i} g_i\right) = \epsilon \sum_i \delta_{g_i} \Gamma(g_i) = \epsilon \sum_i \delta_{g_i} f_{g_i} = f,$$

so Γ is surjective.

Since each pair g_i and f_{g_i} have the same initial and final projections, we have

$$\left\| \sum_{i=1}^n a_i g_i \right\| = \left\| \sum_{i=1}^n a_i f_{g_i} \right\|;$$

thus Γ is isometric. Since Γ is also surjective, it is an automorphism, as desired. $\square$

Notice that we have constructed an automorphism of $\mathcal{B}$ that fixes $\mathcal{D}$ pointwise. By Lemma 3.4 of [V1], such an automorphism is approximately inner.

Applications to Classifications. We outline the application of Theorem 19 to classifying direct limit algebras with particular classes of embeddings. While the refinement, standard, alternation, and twist classifications given in this section are well-known, these proofs seem simpler, in part because of the common framework. In each case, there are two key parts:

(1) the ϕ_i's in the intertwining diagram (5) have a nice form, and
(2) embeddings in the class have a unique factorization.

In Section 8 we give a new classification theorem for algebras with order preserving embeddings. In the next four examples all the finite dimensional algebras are full upper triangular matrix algebras.

Refinement Embedding Limit Algebras. Consider the family of all refinement embeddings ρ_k. Here, k denotes the multiplicity of the embedding while the dimension of the domain algebra is unspecified. Since $\rho_k \circ \rho_l = \rho_{kl} = \rho_l \circ \rho_k$, each refinement embedding can be factored as a composition of refinement embeddings of prime multiplicity, and this factorization is unique up to order.

Given $\varinjlim(A_i, \alpha_i)$ with each α_i a refinement embedding and $A_1 = \mathbb{C}$, we can compute a supernatural number by, for each prime p, counting the number of factors of multiplicity p in the factorization of $\alpha_{1,j}$ and taking the supremum as j goes to infinity.

If $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ are two direct limits of this form and they have the same supernatural numbers as computed above, then it is routine to construct an intertwining diagram such as (5). It follows that the algebras are isometrically isomorphic.

On the other hand, suppose the two algebras are isometrically isomorphic; we will show they have the same supernatural numbers. By Theorem 19, we have an intertwining diagram such as (5). Note that if $\gamma \circ \delta$ is a refinement embedding, then necessarily δ is a refinement embedding, and hence each ϕ_i in the diagram is a refinement embedding.

Since

$$\alpha_{1,m_i} = \phi_{1,2i} \quad \text{and} \quad \beta_{n_1,n_j} = \phi_{2,2j-1},$$

it follows that the supernatural number of $\mathcal{A}$ is given by counting the refinement embeddings of each prime multiplicity in $\phi_1, \phi_2, \ldots$ and the supernatural number of $\mathcal{B}$ is given by counting the number of refinement embeddings of each prime multiplicity in $\beta_{1,n_1}, \phi_2, \phi_3, \ldots$. Since ϕ_1 and β_{1,n_1} are both refinement embeddings from $\mathbb{C}$ to $B_{n_1} = T_x$ for some x, they have the same factorization. Hence the supernatural numbers of $\mathcal{A}$ and $\mathcal{B}$ agree, as required.

This condition is necessary and sufficient, and so classifies this family of algebras.

Standard Embedding Limit Algebras. This classification proceeds in exactly the same way as for refinement embeddings.

Alternation Limit Algebras. Suppose $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ are two direct limits with $A_1 = B_1 = \mathbb{C}$ and each α_i and β_i an alternation embedding (a composition of standard embeddings and refinement embeddings).

Since $\rho_k \circ \sigma_l = \sigma_l \circ \rho_k$, we can factor each alternation embedding as a composition of standard embeddings and refinement embeddings, each of prime multiplicity. Up to ordering, this factorization is unique.

For $\varinjlim(A_i, \alpha_i)$, we can compute two supernatural numbers. For the first, fix a prime p and count the number of standard embeddings of multiplicity p in the unique factorization of $\phi_{1,j}$ for each j, then take the supremum over all j. Repeat this for each prime. For the second, repeat this process, only counting refinement embeddings of multiplicity p instead of standard embeddings. We will also consider the product of these two supernatural numbers, which corresponds to counting all embeddings of each prime multiplicity, standard and refinement.

First, if $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ have their pairs of supernatural numbers agree, each up to a finite factor, and the products of the pair are identical, then the algebras are isometrically isomorphic. Again, the construction of a diagram in the form of (5) is routine.

To show the converse, we need the following fact: If $\alpha \circ \beta$ is an alternation embedding, then β is an alternation embedding. As $\alpha \circ \beta$ is an order preserving embedding, Lemma 1 implies that β is order preserving and so, by Theorem 5, is a direct sum of refinement embeddings. If the summands do not all have the same multiplicity (i.e., β is not an alternation embedding), then the summands in $\alpha \circ \beta$ will not all have the same multiplicity, a contradiction.

Suppose $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ are isometrically isomorphic. Invoking Theorem 19, we have an intertwining diagram of the form of (5). By the previous paragraph, each ϕ_i in the diagram is an alternation embedding. We can compute a pair of supernatural numbers by counting the refinement embeddings of each prime multiplicity and the standard embeddings of each prime multiplicity in the sequence $\phi_1, \phi_2, \ldots$. Since

$$\alpha_{1,m_i} = \phi_{1,2i} \quad \text{and} \quad \beta_{n_1,n_j} = \phi_{2,2j-1},$$

each of the supernatural numbers for $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ can differ by only a finite factor from the supernatural numbers given by the alternation maps ϕ_i.

In particular, each of the supernatural numbers for $\varinjlim(A_i, \alpha_i)$ can differ by only a finite factor from the corresponding number for $\varinjlim(B_i, \beta_i)$. Also, since the product of the two supernatural numbers corresponds to counting all embeddings of a given prime multiplicity, the argument of the previous examples shows that the products must agree exactly. Thus the sufficient condition is also necessary.

Twist Embedding Limit Algebras. A *twist* embedding is an embedding of the form $\operatorname{Ad} U \circ \rho$, where ρ is a refinement embedding and U is the permutation unitary matrix which interchanges the last two minimal projections in the diagonal. In other words, U is the identity matrix with the last two columns interchanged. Limit algebras constructed with these embeddings were first studied in [PePW1]; their classification was given in [HPo]. Unlike the previous examples, the composition of two twist embeddings is not a twist embedding; the natural class to consider consists of embeddings which are compositions of twist embeddings. If $\alpha \colon T_k \longrightarrow T_{nk}$ is a composition of twist embeddings, then it is, in particular, a nest embedding, i.e. it maps invariant projections under T_k to invariant projections for T_{nk}

Quite generally, if ϕ is any embedding from M_k into M_{nk}, then we may write ϕ in the form $\operatorname{Ad} U \circ \rho$ for some permutation unitary U. The choice of U is not unique. Indeed, $\operatorname{Ad} U \circ \rho = \operatorname{Ad} V \circ \rho$ if, and only if, $\operatorname{Ad} V^{-1}U \circ \rho = \rho$; this happens exactly when $V^{-1}U$ is block diagonal with $n \times n$ blocks all of which are equal. Also note that $\operatorname{Ad} U \circ \rho$ is a nest embedding if, and only if, U is block diagonal with each block of size $n \times n$. If we

multiply each block on the right by a fixed $n \times n$ permutation unitary matrix, then the resultant matrix induces the same nest embedding as U does. This allows us to replace U by a matrix in standard form: multiply each block on the right by the inverse of the first block. So, we say that a block diagonal permutation matrix U (with uniform block size) is in *standard form* if the first block is the identity matrix. One other trivial fact about nest embeddings should be noted: if $\phi \circ \psi$ is a nest embedding, then ψ must be a nest embedding.

Suppose, now, that $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ are two direct limit algebras with each α_i and β_i a twist embedding. By Theorem 19, we have an intertwining diagram as in (7). For each i, $\phi_{i+1} \circ \phi_i$ is a nest embedding; consequently, each ϕ_i is a nest embedding. We shall show below that each ϕ_i is a composition of twist embeddings and further, that each composition of twist embeddings has only one factorization into twist embeddings. From this we can conclude that there is some $m \leq m_1$ such that $A_{m+i} = B_{n_1+i}$ and $\alpha_{m+i} = \beta_{n_1+i}$, for all i. This necessary condition for isomorphic isomorphism is clearly also sufficient.

In order to verify the second of the two claims above, suppose that $\mathrm{Ad}\, V \circ \rho$ is a composition of twist embeddings, where V is in standard form. The general observation: $\rho \circ \mathrm{Ad}\, U = (\mathrm{Ad}\, \rho(U)) \circ \rho$ can then be used to see that $V = U_q \circ \cdots \circ U_1$ is a product of permutation unitaries each of which is the image of an identity matrix with the last two columns interchanged under a refinement embedding of suitable multiplicity. The critical observation is that it is possible to read off from the matrix V, which is uniquely determined by the requirement that it be in standard form, the multiplicities of the refinements which are applied to the U_i. This yields a unique factorization for the original embedding $\mathrm{Ad}\, V \circ \rho$ as a composition of twist embeddings.

To verify the other claim, assume that $\tau = \mathrm{Ad}\, V \circ \rho_p = \nu \circ \mu$, where τ is a composition of twist embeddings, V is a permutation unitary in standard form, ρ_p is a refinement of multiplicity p defined on some T_k, and ν and μ are nest embeddings with multiplicities n and m respectively. We need to prove that ν and μ are actually compositions of twist embeddings. Since ν and μ are nest embeddings, there are unique permutation unitaries V_n and V_m in standard form so that $\nu = \mathrm{Ad}\, V_n \circ \rho_n$ and $\mu = \mathrm{Ad}\, V_m \circ \rho_m$. Here, ρ_n and ρ_m are refinement embeddings of multiplicities n and m and, of course, $p = nm$. From the uniqueness of standard form and the general observation in the preceding paragraph, it is easy to see that $V = V_n \rho_n(V_m)$. In order for V to have the form which the standard permutation unitary associated with a composition of twist embeddings must have, it is necessary that both V_n and V_m also have these forms. This means that ν and μ are compositions of twists.

Ordered Bratelli diagrams with multiplicity. As a final example, consider algebras $\varinjlim(A_i, \alpha_i)$ where each α_i is order preserving and each A_i is a direct sum of T_n's. These algebras can be described in terms of ordered Bratelli diagrams, introduced by Power in [Po5]. We begin by recalling the definition of ordered Bratelli diagram given in [PW]. The

definitions of ordered diagram and ordered diagram with multiplicity have been given in Section 2.

Definition. An *ordered Bratelli diagram* is a pair $(\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = V_0 \cup V_1 \cup \cdots ,$$

a disjoint union of finite sets with V_0 a singleton, and

$$\mathcal{E} = \{(E_n, r_n, s_n) \mid n \geq 1\}$$

where each (E_n, r_n, s_n) is an ordered diagram from V_{n-1} to V_n.

By *ordered Bratelli diagram with multiplicity*, we mean an ordered Bratelli diagram as above with each (E_n, r_n, s_n) replaced by (E_n, r_n, s_n, f_n), an ordered diagram with multiplicity.

Using Theorem 6, we can associate an ordered Bratelli diagram with multiplicity to each unital triangular AF algebra $\varinjlim(A_i, \alpha_i)$ where each α_i is order preserving, each A_i is a direct sum of T_n's, and $A_0 = \mathbb{C}$. We describe such triangular AF algebras in terms of their ordered Bratelli diagrams with multiplicity. This extends, in a natural way, Theorem 3.7 of [PW] where standard $\mathbb{Z}$-analytic TAF algebras are classified by their associated ordered Bratelli diagrams.

First, we define the analogue of composition for ordered diagrams with multiplicity, following [PW].

Definition. Given two ordered diagrams with multiplicity, (E_1, r_1, s_1, f_1) from V_1 to V_2 and (E_2, r_2, s_2, f_2) from V_2 to V_3, their *contraction* is an ordered diagram with multiplicity (E, r, s, f) from V_1 to V_3 given by

$$E = \{(e_1, e_2) \in E_1 \times E_2 \mid r_1(e_1) = s_2(e_2)\},$$

and

$$s(e_1, e_2) = s_1(e_1), \quad r(e_1, e_2) = r_2(e_2), \text{ and } f(e_1, e_2) = f_1(e_1) f_2(e_2).$$

Given two edges, (e_1, e_2) and (f_1, f_2), with $r(e_1, e_2) = r(f_1, f_2)$ then $r_2(e_2) = r_2(f_2)$. If $e_2 \neq f_2$, then order (e_1, e_2) and (f_1, f_2) as e_2 and f_2 are ordered; if $e_2 = f_2$, then $r_1(e_1) = r_1(f_1)$ and we can order (e_1, e_2) and (f_1, f_2) as e_1 and f_1 are ordered.

We denote E as $E_2 \circ E_1$.

As the notation suggests, if ϕ_1 and ϕ_2 are embeddings associated to E_1 and E_2, then $\phi_2 \circ \phi_1$ is the embedding associated to $E_2 \circ E_1$. It follows that $(E_3 \circ E_2) \circ E_1 = E_3 \circ (E_2 \circ E_1)$, although this is also trivial to show directly.

Again following [HPS] and [PW], we have:

Definition. Given two ordered Bratelli diagrams with multiplicity, $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{W}, \mathcal{F})$, we say they are *order equivalent* if there exist strictly increasing functions $f, g \colon \mathbb{N} \to \mathbb{N}$ and ordered diagrams with multiplicity E'_n from V_n to $W_{f(n)}$ and F'_n from W_n to $V_{g(n)}$ so that

$$F'_{f(n)} \circ E'_n \cong^{ord} E_{g(f(n))} \circ \cdots \circ E_{n+1} \tag{6}$$

and

$$E'_{g(n)} \circ F'_n \cong^{ord} F_{f(g(n))} \circ \cdots \circ F_{n+1} \tag{7}$$

for all $n \in \mathbb{N}$.

It is now routine to prove:

Theorem 20. *Suppose $\mathcal{A} = \varinjlim(A_i, \alpha_i)$ and $\mathcal{B} = \varinjlim(B_i, \beta_i)$ are unital triangular AF algebras with each A_i and B_i a direct sums of T_n's, $A_0 = B_0 = \mathbb{C}$, and each α_i and β_i order preserving.*

There is an isometric isomorphism $\Phi \colon \mathcal{A} \to \mathcal{B}$ if, and only if, the ordered Bratelli diagrams with multiplicity associated to $\varinjlim(A_i, \alpha_i)$ and $\varinjlim(B_i, \beta_i)$ are order equivalent.

Proof. Suppose there is an isometric isomorphism $\Phi \colon \mathcal{A} \to \mathcal{B}$. By Theorem 19, we have the following commuting diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_{1,m_1}"] \arrow[dr, "\phi_1"] & A_{m_1} \arrow[r, "\alpha_{m_1,m_2}"] \arrow[dr, "\phi_3"] & A_{m_2} \arrow[r, "\alpha_{m_2,m_3}"] \arrow[dr, "\phi_5"] & A_{m_3} \arrow[r, "\alpha_{m_3,m_4}"] \arrow[dr, "\phi_7"] & A_{m_4} \arrow[r, "\alpha_{m_5,m_4}"] \arrow[dr, "\phi_9"] & \cdots \mathcal{A} \arrow[d, leftrightarrow, "{\Phi,\Phi^{-1}}"] \\
B_1 \arrow[r, "\beta_{1,n_1}"] & B_{n_1} \arrow[r, "\beta_{n_1,n_2}"] \arrow[u, "\phi_2"] & B_{n_2} \arrow[r, "\beta_{n_2,n_3}"] \arrow[u, "\phi_4"] & B_{n_3} \arrow[r, "\beta_{n_3,n_4}"] \arrow[u, "\phi_6"] & B_{n_4} \arrow[r, "\beta_{n_4,n_5}"] \arrow[u, "\phi_8"] & \cdots \mathcal{B}.
\end{tikzcd} \tag{8}$$

By Lemma 1, each ϕ_i is order preserving, since $\phi_{i+1} \circ \phi_i$ equals either some $\alpha_{a,b}$ or some $\beta_{a,b}$. By Theorem 6, we can associate an ordered diagram with multiplicity to each ϕ_i, say P_i.

Let $(\mathcal{V}, \mathcal{E})$ be the ordered Bratelli diagram with multiplicity associated to $\varinjlim(A_i, \alpha_i)$ and $(\mathcal{W}, \mathcal{F})$ be the one associated to $\varinjlim(B_i, \beta_i)$. To show they are order equivalent, we define $f, g \colon \mathbb{N} \to \mathbb{N}$ by $f(k) = n_j$ where j is the least integer with $k \leq m_{j-1}$ and by $g(k) = m_j$ where j is the least integer with $k \leq n_j$. Define E'_k to be $P_{2j+1} \circ X$ where X is the ordered diagram with multiplicity associated to α_{k,m_j} and j is the least integer with $k \leq m_j$. Similarly, F'_k is $P_{2j} \circ X$ where X is the order diagram with multiplicity associated to β_{k,n_j} and j is the least integer with $k \leq n_j$. Commutativity of the diagram implies that (6) and (7) hold.

Conversely, if the diagrams are order equivalent, we can construct sequences $\{m_i\}$ and $\{n_j\}$ and embeddings ϕ_i so that the diagram (8) commutes. It follows immediately that there is an isometric isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Choose n_1 to be $f(1)$, m_1 to be $g(n_1)$, and for $i > 1$ choose n_i to be $f(m_{i-1})$ and m_i to be $g(n_i)$. Define ϕ_1 to be the embedding associated to the ordered diagram with multiplicity E'_1, and for $i > 1$ define ϕ_{2i} to be the embedding associated to F'_{n_i} and ϕ_{2i+1} to be the embedding associated to E'_{m_i}. Now (6) and (7) imply that the diagram (8) commutes. $\square$

Unlike the previous examples, here we have found no way to pick a canonical representative from an equivalence class of isometrically isomorphic algebras. The difficulty is that we do not have a unique factorization theorem for order preserving embeddings between direct sums of T_n's. Thus, Theorem 20 is a variant of Theorem 19 rather than a true classification theorem. In the next section, we restrict to order preserving embeddings between T_n's and obtain a unique factorization theorem. Such a theorem is crucial for the classification given in the final section.

7 Unique Factorization for Order Preserving Embeddings

In the last section, our aim was to demonstrate that factorization theorems for families of embeddings yield necessary and sufficient conditions for classification. In this section, we prove unique factorization theorems for order preserving embeddings. While the proof is somewhat technical and requires several preliminary lemmas, the statement of the factorization, Theorem 27, is simple. In the next section, we will use this factorization to classify limit algebras with order preserving presentations.

By Theorem 5, an embedding between T_n's is order preserving if, and only if, it is a direct sum of refinement embeddings. For the sake of brevity, we use $(a_0, \ldots, a_{n-1})$ to denote the direct sum of n refinement embeddings with multiplicities $a_0, a_1, \ldots, a_{n-1}$ respectively, that is, $\rho_{a_0} \oplus \rho_{a_1} \oplus \cdots \oplus \rho_{a_{n-1}}$. We refer to the number of entries in a tuple as its length, and denote the length of a by $\operatorname{len} a$.

With this notation, the composition of two embeddings, say $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{m-1})$, is an mn-tuple. Notice that $a \circ b$ equals

$$(a_0b_0, a_0b_1, \ldots, a_0b_{m-1}, a_1b_0, \ldots, a_1b_{m-1}, a_2b_0, \ldots, a_{n-2}b_{m-1}, a_{n-1}b_0, \ldots, a_{n-1}b_{m-1})$$

while in the other order, $b \circ a$ equals

$$(b_0a_0, b_0a_1, \ldots, b_0a_{n-1}, b_1a_0, \ldots, b_1a_{n-1}, b_2a_0, \ldots, b_{m-2}a_{n-1}, b_{m-1}a_0, \ldots, b_{m-1}a_{n-1}).$$

Clearly, refinement embeddings commute with all embeddings.

Consider some tuple $a = (a_0, \ldots, a_{n-1})$ with integer entries. Dividing by a_0 gives a new, normalized tuple $b = (1, b_1, \ldots, b_{n-1})$ with rational entries. Because of Theorem 13, we need only consider normalized tuples; i.e., tuples with rational entries and first entry always 1.

One advantage of this standard form is that we immediately have the following lemma:

Lemma 21. *If* $b \circ c = (1, a_1, \dots, a_{n-1})$, *and* $\operatorname{len} c = m$, *then* $c = (1, a_1, \dots, a_{m-1})$.

Hence if $b \circ c = d \circ e$ *and* $\operatorname{len} c = \operatorname{len} e$, *then* $c = e$ *and* $b = d$.

Proof. That $c = (1, a_1, \dots, a_{m-1})$ is immediate from the expression for composition. It is obvious that $c = e$, while $b = d$ follows from $c = e$ and the expression for composition. $\square$

Consider some tuple $a = (1, a_1, \dots, a_{n-1})$. Given an integer m, we say a is *m-divisible* if m divides n and the ratios a_i/a_{i-1} and a_j/a_{j-1} are equal for all i and j such that $i \equiv j \not\equiv 0 \pmod m$. We will say a is *strongly m-divisible* if m divides n and the ratios a_i/a_{i-1} and a_j/a_{j-1} are equal for all i and j such that $i \equiv j \pmod m$. Notice that if $a = (1, x, x^2, x^3, \dots x^{n-1})$ then a will be strongly m-divisible for m any factor of n; in particular, a is strongly 1-divisible if, and only if, a is a geometric sequence.

Lemma 22. *Consider a tuple* $a = (1, a_1, \dots, a_{n-1})$ *and an integer* m *such that* $1 < m < n$.

Then there is a tuple c such that $a = b \circ c$ *with* $\operatorname{len} c = m$ *if, and only if, a is m-divisible. In this case,* $c = (1, a_1, \dots, a_{m-1})$ *and* $b = (1, a_m, a_{2m}, a_{3m}, \dots, a_{n-m})$.

Further $a = b \circ c$ *with* $\operatorname{len} c = m$ *and b a geometric sequence if, and only if, a is strongly m-divisible. In this case,* $c = (1, a_1, \dots, a_{m-1})$ *and* $b = (1, a_m, a_m^2, a_m^3, \dots, a_m^{n/m-1})$.

Proof. Let $k = n/m$. We begin with m-divisibility.

If $a = b \circ c$ where c is an m-tuple, then clearly m divides n. For any l such that $0 \le l < k$, we have $(a_{ml}, a_{ml+1}, \dots, a_{ml+(m-1)})$ is a_{ml} times c, that is $a_{ml}(1, a_1, \dots, a_{m-1})$. Thus a is m-divisible.

If in addition, b is a geometric sequence, then

$$\frac{a_{m(l+1)}}{a_{ml+(m-1)}} = \frac{a_{m(l+1)}}{a_{ml}a_{m-1}} = \frac{a_m^{l+1}}{a_m^l a_{m-1}} = \frac{a_m}{a_{m-1}},$$

so a is strongly m-divisible.

Conversely, suppose a is m-divisible and let b and c be as in the lemma. If $i = qm + r$, with $0 \le r < m$, then the i^{th} entry of $b \circ c$ is $b_q c_r$. We will prove $b_q c_r = a_i$ by induction on r. If $r = 0$, then $b_q c_0 = a_{qm} 1 = a_i$. If the result holds for $r - 1$, then $a_{i-1} = b_q c_{r-1}$. Now as $i \equiv r \not\equiv 0 \pmod m$, we have that $a_i/a_{i-1} = a_r/a_{r-1}$. However by the definition of c, $a_r/a_{r-1} = c_r/c_{r-1}$, so

$$a_i = a_{i-1}\frac{a_i}{a_{i-1}} = (b_q c_{r-1})\frac{a_r}{a_{r-1}} = b_q\left(c_{r-1}\frac{c_r}{c_{r-1}}\right) = b_q c_r.$$

Thus, $a = b \circ c$.

Strong m-divisibility implies that, for $j = 1, 2, \dots, k-1$,

$$\frac{a_{jp}}{a_{(j-1)p+p-1}} = \frac{a_p}{a_{p-1}}$$

and since $a_{(j-1)p+p-1} = a_{(j-1)p}a_{p-1}$, we have $a_{jp} = a_p a_{(j-1)p}$ for $j = 1, \dots, k-1$. Thus $a_{jp} = a_p^j$, as required. $\square$

Lemma 23. *Let p and q be positive integers. Suppose $a = (1, a_1, \dots, a_{n-1})$ is p-divisible and q-divisible. Then $a = b \circ c$ where $\operatorname{len} c = \operatorname{lcm}(p, q)$ and c is strongly $\gcd(p, q)$-divisible.*

Proof. That a can be factored as $b \circ c$ with $\operatorname{len} c = \operatorname{lcm}(p, q)$ is immediate from Lemma 22. We need only show $c = (1, a_1, \dots, a_{m-1})$ is strongly $\gcd(p, q)$-divisible.

Suppose $i \in \{1, \dots, p-1\}$ and $j \in \{1, \dots, q-1\}$ such that $i \equiv j \pmod{\gcd(p, q)}$. It follows from the Chinese remainder theorem that there is a unique integer r less than $\operatorname{lcm}(p, q)$ so that $r \equiv i \pmod p$ and $r \equiv j \pmod q$. The p-divisibility and q-divisibility imply that $a_i/a_{i-1} = a_r/a_{r-1} = a_j/a_{j-1}$. From this, it is easy to show that if $i \equiv j \pmod{\gcd(p, q)}$ and either $i, j \in \{1, \dots, q-1\}$ or $i, j \in \{1, \dots, p-1\}$, then $a_i/a_{i-1} = a_j/a_{j-1}$.

Turning to the general case, suppose r and s are integers so that $0 < r, s < \operatorname{lcm}(p, q)$ and $r \equiv s \pmod{\gcd(p, q)}$. There are integers i_r and j_r such that $0 \le i_r < p$ with $i_r \equiv r \pmod p$ and $0 \le j_r < q$ with $j_r \equiv r \pmod q$. As $0 < r < \operatorname{lcm}(p, q)$, at least one of i_r or j_r is nonzero. Similar statements hold for s. Thus, one of the four pairs

$$(i_r, j_s), \quad (j_r, i_s), \quad (i_r, i_s), \quad (j_r, j_s)$$

must have both entries nonzero. Let (k, l) be that pair; using p and q-divisibility, we have $a_r/a_{r-1} = a_k/a_{k-1}, a_s/a_{s-1} = a_l/a_{l-1}$, and by the previous paragraph, $a_k/a_{k-1} = a_l/a_{l-1}$. Thus, $a_r/a_{r-1} = a_s/a_{s-1}$ and we are done. $\square$

Definition. Call $a = (1, a_1, \dots, a_{n-1})$ *irreducible* if a cannot be factored nontrivially.

Lemma 24. *Given a tuple $a = (1, a_1, \dots, a_{n-1})$, there is a minimal integer m so that $a = b \circ c$ with $\operatorname{len} c = m$. Further, $c = (1, a_1, \dots, a_{m-1})$ and c is irreducible.*

Proof. Choose the least positive integer m, $1 < m \le n$, so that a is m-divisible. If $m = n$ then a is irreducible and we take $c = a$, $b = (1)$. If $m < n$ then by Lemma 22 we can factor a as $b \circ c$ with $\operatorname{len} c = m$.

If c were reducible, this would contradict the minimality of m. By Lemma 21, $c = (1, a_1, \dots, a_{m-1})$ and we are done. $\square$

Geometric sequences do not have a unique factorization into irreducibles; for example, $(1, x, x^2, x^3, x^4, x^5, x^6)$ can be written as either $(1, x^3) \circ (1, x, x^2)$ or $(1, x^2, x^4) \circ (1, x)$. In general, if a is a geometric sequence and $a = b \circ c$, then both b and c will be geometric sequences and the ratio of b is exactly the ratio of c raised to the power $\operatorname{len} c$. It follows that for a tuple which is a geometric sequence, if we factor its length into primes then for each distinct ordering of these primes there will be a distinct factorization of the tuple. There are two solutions to this problem: either we order such factors according to length, or we avoid factoring such tuples at all. We will give a factorization theorem for each of these solutions.

First we need a technical lemma.

Lemma 25. *Suppose x, y, z and w are tuples with y and w irreducible and* $\operatorname{len} y \neq \operatorname{len} w$.

If $x \circ y = z \circ w$, then y is a geometric sequence and $x = z' \circ w'$ where w' is an irreducible geometric sequence so that $\operatorname{len} w' = \operatorname{len} w$ *and $w' \circ y$ is a geometric sequence.*

Proof. Let $m = \operatorname{len} y, n = \operatorname{len} w$, and $a = x \circ y$. By hypothesis, $m \neq n$.

Notice that as a is m-divisible and n-divisible, it is also $\gcd(m, n)$-divisible. Since y and w are initial segments of a and $\gcd(m, n) < \max\{m, n\}$, if $\gcd(m, n) > 1$ then at least one of y or w is reducible, a contraction.

Thus $\gcd(m, n) = 1$ and so by Lemma 23 we may conclude that $a = e \circ f$ where $\operatorname{len} f = mn$ and f is a geometric sequence. Also, $f = f_2 \circ f_1$ where f_1 and f_2 are geometric sequences, $\operatorname{len} f_2 = n$ and $\operatorname{len} f_1 = m$. Since $e \circ f_2 \circ f_1 = x \circ y$ and $\operatorname{len} f_1 = \operatorname{len} y$, Lemma 21 shows $f_1 = y$ and $x = e \circ f_2$.

In particular, y is a geometric sequence and if we let $z' = e$ and $w' = f_2$ then $w' \circ y$ is a geometric sequence. Also, $\operatorname{len} w' = n = \operatorname{len} w$, as required. To show w' is irreducible, we must first note that since y is an irreducible geometric sequence, $\operatorname{len} y$ must be a prime. Since our hypothesis are symmetric in w and y, similarly $\operatorname{len} w$ is also a prime and so w' is irreducible. □

Theorem 26. *Every tuple has a unique factorization of the form $c_1 \circ c_2 \circ \cdots \circ c_k$, where each c_i is an irreducible tuple such that if $c_i \circ c_{i+1}$ is a geometric sequence, then* $\operatorname{len} c_i \geq \operatorname{len} c_{i+1}$.

Proof. By repeatedly applying Lemma 24, we can factor a into irreducibles and we claim that this factorization has the given form. Suppose, for some i, c_i and c_{i+1} are geometric sequences for which $c_i \circ c_{i+1}$ is also a geometric sequence. Let $n = \operatorname{len} c_i$ and $m = \operatorname{len} c_{i+1}$. We must show $n \geq m$.

Notice $c_i \circ c_{i+1} = e \circ f$ where $f = (1, x, x^2, \ldots, x^{n-1})$ and $e = (1, x^n, x^{2n}, \ldots, x^{(m-1)n})$. Thus $c_1 \circ c_2 \circ \ldots \circ c_{i+1} = c_1 \circ c_2 \circ \ldots c_{i-1} \circ e \circ f$. However, when we factored $c_1 \circ c_2 \circ \ldots \circ c_{i+1}$ using Lemma 24, we chose c_{i+1} as the factor of minimal length. Hence $n = \operatorname{len} f \geq m$, as required.

It remains only to show that this factorization is unique. Suppose a can be factored in the above form as $c_1 \circ c_2 \circ \cdots \circ c_k$ and as $d_1 \circ d_2 \circ \cdots \circ d_l$. If $\operatorname{len} c_k = \operatorname{len} d_l$, then by Lemma 21, $c_k = d_l$ and $c_1 \circ \cdots \circ c_{k-1} = d_1 \circ \cdots \circ d_{l-1}$. By induction we are done, in this case.

Otherwise we have $\operatorname{len} c_k \neq \operatorname{len} d_l$ and will get a contradiction by showing $\operatorname{len} c_k = \operatorname{len} d_l$. By symmetry, it suffices to prove that $\operatorname{len} d_l \geq \operatorname{len} c_k$.

Since $\operatorname{len} c_k \neq \operatorname{len} d_l$, we can apply Lemma 25 with

$$x = c_1 \circ \cdots \circ c_{k-1}, \quad y = c_k, \quad z = d_1 \circ \cdots \circ d_{l-1}, \text{ and } w = d_l.$$

Thus c_k is a geometric sequence and $c_1 \circ \cdots \circ c_{k-1} = z_1 \circ w_1$ with $\operatorname{len} w_1 = \operatorname{len} d_l$ and w_1 an irreducible geometric sequence so that $w_1 \circ c_k$ is a geometric sequence.

If $\operatorname{len} c_{k-1} = \operatorname{len} w_1$, then by Lemma 21 $c_{k-1} = w_1$. Thus $c_{k-1} \circ c_k$ is a geometric sequence. By the assumed form of $c_1 \circ \cdots \circ c_k$, we can conclude $\operatorname{len} c_{k-1} \geq \operatorname{len} c_k$ and so $\operatorname{len} d_l = \operatorname{len} c_{k-1} \geq \operatorname{len} c_k$, as required.

Otherwise, $\operatorname{len} c_{k-1} \neq \operatorname{len} w_1$ so by Lemma 25 with

$$x = c_1 \circ \cdots \circ c_{k-2}, \quad y = c_{k-1}, \quad z = z_1, \text{ and } w = w_1,$$

we have c_{k-1} is a geometric sequence and $c_1 \circ \cdots \circ c_{k-2} = z_2 \circ w_2$, where $\operatorname{len} w_2 = \operatorname{len} w_1$ and w_2 an irreducible geometric sequence so that $w_2 \circ c_{k-1}$ is a geometric sequence.

Further, since w_1 is a geometric sequence and $c_1 \circ \cdots \circ c_{k-1} = z_1 \circ w_1$ we can conclude that the ratio of w_1 equals the ratio of c_{k-1}. As the ratio of c_{k-1} equals the ratio of $w_2 \circ c_{k-1}$, and $w_1 \circ c_k$ is a geometric sequence, it follows that $w_2 \circ c_{k-1} \circ c_k$ is a geometric sequence.

If $\operatorname{len} c_{k-2} = \operatorname{len} w_2$, then $c_{k-2} = w_2$. As before with c_{k-1} and w_1, we may conclude that $c_{k-2} \circ c_{k-1} \circ c_k$ is a geometric sequence and $\operatorname{len} d_l = \operatorname{len} c_{k-2} \geq \operatorname{len} c_{k-1} \geq \operatorname{len} c_k$, as required. If $\operatorname{len} c_{k-2} \neq \operatorname{len} w_2$, then we can apply Lemma 25 again.

Continuing in this way, we will either prove $\operatorname{len} d_l \geq \operatorname{len} c_k$ or end up with $c_1 = z_{k-1} \circ w_{k-1}$ where $\operatorname{len} w_{k-1} = \operatorname{len} d_l$. Since c_1 is irreducible, it follows that $z_{k-1} = (1)$ and so $w_{k-1} = c_1$. As above, $c_1 \circ \cdots \circ c_k$ is a geometric sequence and so $\operatorname{len} c_1 \geq \operatorname{len} c_2 \geq \cdots \geq \operatorname{len} c_k$. Thus, $\operatorname{len} d_l \geq \operatorname{len} c_k$, as required. □

It is clear that in the above factorization we can repeatedly multiply together adjacent factors whose products will be geometric sequences to get:

Theorem 27. *Every tuple has a unique factorization of the form $c_1 \circ c_2 \circ \cdots \circ c_k$ satisfying:*

a) *each c_i is either an irreducible tuple or a geometric sequence, and*
b) *for all $i < k$, $c_i \circ c_{i+1}$ is not a geometric sequence.*

Finally, we consider when two embeddings commute:

Corollary 28. *Suppose a and b are normalized tuples and $a \circ b = b \circ a$. Then one of the following holds:*

(1) *either $a = (1)$ or $b = (1)$,*
(2) *$a = (1, \ldots, 1)$ and $b = (1, \ldots, 1)$, with possibly unequal lengths,*
(3) *there exist a tuple c and integers $m, n \in \mathbb{N}$ so that $a = c^m$ and $b = c^n$.*

Remark. In terms of commutativity of embeddings, condition (1) states that a refinement embedding commutes with any order preserving embedding. Condition (2) states that any two standard embeddings commute.

Proof. Let $d = a \circ b = b \circ a$. If we are not in the third case, then it follows that d has two factorizations: one given by factoring $a \circ b$ and another given by factoring $b \circ a$. We can conclude by Theorem 26 that d, a and b are geometric series.

Further, $d = a \circ b$ implies the ratio of a is the ratio of b raised to the power $\operatorname{len} b$. On the other hand $d = b \circ a$ implies the ratio of b is the ratio of a raised to the power $\operatorname{len} a$. If $\operatorname{len} a = 1$ or $\operatorname{len} b = 1$, then we are in the first case. Otherwise, the ratios of a and b are both 1, which is precisely the second case. □

8 Classification of Order Preserving Presentations

In this section, we classify all limit algebras which have presentations, $\varinjlim(T_{n_i}, \alpha_i)$ with each α_i an order preserving embedding. By Theorem 5, each α_i is a direct sum of refinement embeddings. Let $\alpha_{x,y} = \alpha_{y-1} \circ \cdots \circ \alpha_{x+1} \circ \alpha_x$ for all integers x and y with $1 \leq x < y$. As in the last section, we can identify each $\alpha_{x,y}$ with a tuple, the finite sequence of refinement multiplicities.

Definition. We say that an order preserving presentation of a limit algebra $\mathcal{A}$ has *geometric character* if there is some N so that for all m and n larger than N, the tuple associated to $\alpha_{m,n}$ is a geometric sequence. (We shall show that this is well-defined for the limit algebra $\mathcal{A}$ itself in the course of proving the classification theorem. Consequently, when $\mathcal{A}$ has an order preserving presentation with geometric character, we say that $\mathcal{A}$ has geometric character.)

Geometric character implies that, for sufficiently large m, the order preserving embedding $\alpha_{1,m}$ factors as a finite sequence (not depending on m) followed by a geometric sequence whose length depends on m. Note that the ratio of this geometric sequence depends on the choice of the finite sequence but not on m.

Choose the initial segment and consider the geometric sequences that follow it. Since the product of an m-tuple and an n-tuple is an mn-tuple, the length of the geometric sequence in $\alpha_{1,m}$ divides the length of the geometric sequence in $\alpha_{1,m+1}$. Thus, we can associate a supernatural number to this sequence of lengths by counting the number of times a given prime divides any length in the sequence.

By the *reduced root* of a rational number, q, we mean the rational number $q^{1/n}$ such that for all $m > n$, $q^{1/m}$ is not a rational number.

Two supernatural numbers, a and b, are *finitely equivalent* if there are finite integers m and n so that ma and nb are the same supernatural number. In other words, if $a(p)$ and $b(p)$ are the exponents for the prime p in a and b respectively, then $a(p) \neq b(p)$ for only finitely many p and only when both $a(p)$ and $b(p)$ are both finite. Similarly, two unique factorizations of sequences of normalized tuples are *finitely equivalent* if either factorization can be converted to the other by changing only finitely many factors.

If $\mathcal{A}$ has geometric character, the invariants are:

1. the supernatural number of the C*-envelope (a UHF C*-algebra)
2. the finite equivalence class of the supernatural number of the first summands
3. the finite equivalence class of the supernatural number of the lengths
4. the reduced root of the ratio of the geometric sequence

If $\mathcal{A}$ does not have geometric character, the invariants are:

(1) the supernatural number of the C*-envelope (again a UHF C*-algebra)
(2) the finite equivalence class of the supernatural number of the first summands
(3) the finite equivalence class of the unique factorization of the sequence of normalized tuples

Remark. Alternation algebras are a special subcase of the geometric character case. Each normalized tuple in the presentation for an alternation algebra has all entries equal to 1. The lengths of the tuples are exactly the multiplicities of the standard embedding factors. Thus invariant (2) is just the finite equivalence class of the refinement multiplicities, invariant (3) is the finite equivalence class of the standard multiplicities, and invariant (4) is necessarily equal to 1.

Theorem 29. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are triangular AF algebras and there is an isometric isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$.*

If $\mathcal{A}$ has a presentation, $\varinjlim(T_{n_i}, \alpha_i)$, with each α_i order preserving, then so does $\mathcal{B}$ and either they both have geometric character or they both don't. In either case, they have the same invariants.

Conversely, two such algebras with the same invariants are isometrically isomorphic.

Proof. Suppose that $\mathcal{A}$ has a presentation $\varinjlim(A_i, \alpha_i)$ with each A_i the upper triangulars of some full matrix algebra and each α_i a direct sum of refinement embeddings. It is straightforward to see that $\mathcal{B}$ has an order preserving presentation, since $\varinjlim(A_i, \alpha_i)$ is also a presentation for $\mathcal{B}$. To see this formally, suppose for each i, $\theta_i\colon A_i \to \mathcal{A}$ is the isomorphism between A_i and the isomorphic subalgebra of $\mathcal{A}$. The embedding $\Phi \circ \theta_i\colon A_i \to \mathcal{B}$ gives a subalgebra of $\mathcal{B}$ isomorphic to A_i, for each i.

Suppose $\mathcal{B}$ has some other presentation $\varinjlim(B_i, \beta_i)$ with each B_i the upper triangulars of some full matrix algebra and each β_i a direct sum of refinement embeddings. We must show that this presentation of $\mathcal{B}$ has geometric character if and only if the presentation of $\mathcal{A}$ does.

By Theorem 19, we have an intertwining diagram:

$$
\begin{tikzcd}
A_1 \arrow[r, "\alpha_{1,m_1}"] \arrow[dr, "\phi_1"] & A_{m_1} \arrow[r, "\alpha_{m_1,m_2}"] \arrow[dr, "\phi_3"] & A_{m_2} \arrow[r, "\alpha_{m_2,m_3}"] \arrow[dr, "\phi_5"] & A_{m_3} \arrow[r, "\alpha_{m_3,m_4}"] \arrow[dr, "\phi_7"] & A_{m_4} \arrow[r, "\alpha_{m_5,m_4}"] \arrow[dr, "\phi_9"] & \cdots \mathcal{A} \arrow[d, leftrightarrow, "{\Phi,\Phi^{-1}}"] \\
& B_{n_1} \arrow[u, "\phi_2"] \arrow[r, "\beta_{n_1,n_2}"] & B_{n_2} \arrow[u, "\phi_4"] \arrow[r, "\beta_{n_2,n_3}"] & B_{n_3} \arrow[u, "\phi_6"] \arrow[r, "\beta_{n_3,n_4}"] & B_{n_4} \arrow[u, "\phi_8"] \arrow[r, "\beta_{n_4,n_5}"] & \cdots \mathcal{B}
\end{tikzcd} \tag{9}
$$

If the presentation of $\mathcal{A}$ has geometric character, we may choose the sequence $\{m_i\}$ so that α_{m_i,m_j} is a geometric sequence for each i and j with $1 \le i < j$. It is straightforward to observe that the product of two tuples is a geometric sequence only if both of the original tuples are geometric sequences. Since $\alpha_{m_i,m_j} = \phi_{2j} \circ \beta_{n_{i+1},n_j} \circ \phi_{2i+1}$, it follows that $\beta_{j,k}$

is a geometric sequence for all j and k with $k > j \geq n_2$. Thus, the presentation of $\mathcal{B}$ has geometric character.

Similarly, if the presentation of $\mathcal{B}$ has geometric character, then so does the presentation of $\mathcal{A}$.

We now prove that $\mathcal{A}$ and $\mathcal{B}$ have the same invariants.

Using Theorem 7.5 of [Po4] we can extend Φ to a $*$-isomorphism between the C*-envelopes of $\mathcal{A}$ and $\mathcal{B}$. Since the C*-envelopes are UHF C*-algebras, by Glimm's classification the supernatural numbers of the C*-envelopes must agree.

We have already proved, in Theorem 13, that the supernatural numbers of the first refinement summands must agree up to finite equivalence. We now divide all tuples through by their first entry, so that all tuples begin with 1. This allows us to apply the unique factorization theorem of the last section, Theorem 27.

Case 1: $\mathcal{A}$ has geometric character. From the diagram (9), we have

$$\alpha_{m_1,m_j} = \phi_{2j} \circ \beta_{n_2,n_j} \circ \phi_3. \tag{10}$$

As above, we may assume that α_{m_1,m_j} is a geometric sequence for each $j > 1$. It follows that β_{n_2,n_j} is also a geometric sequence for each $j \geq 2$.

Recall that for any tuples a and b, the product $b \circ a$ is a geometric sequence only if both a and b are geometric sequences with the ratio of b equal to the ratio of a raised to the power len b. Thus the ratio of β_{n_2,n_j} is a power of the ratio of α_{m_1,m_j}, and therefore a power of the reduced root of the presentation for $\mathcal{A}$.

Observe that $\beta_{1,n_j} = \beta_{n_2,n_j} \circ \beta_{1,n_2}$ will factor as some initial segment followed by a geometric sequence. The ratio of this geometric sequence will be a rational number that is a root of the ratio of β_{n_2,n_j}. In particular, we can conclude that the reduced roots given by these two presentations of $\mathcal{A}$ and $\mathcal{B}$ must be the same.

Also from (10), we can conclude that, for each $j > 2$, the length of the geometric sequence β_{n_2,n_j} must divide the length of the geometric sequence α_{m_1,m_j}. It follows that, after deletion of a finite factor, the supernatural number associated with the lengths of the geometric sequences for $\mathcal{B}$ divides the corresponding supernatural number for $\mathcal{A}$. Since we may interchange the roles of $\mathcal{A}$ and $\mathcal{B}$, we have that, up to finite equivalence, the supernatural numbers associated with the lengths agree. This completes case 1.

Case 2: $\mathcal{A}$ does not have geometric character. Since $\mathcal{A}$ does not have geometric character, for any integer y there is an integer $z > y$ so that the (unique) factorization of $\alpha_{y,z}$ is not a geometric sequence. Hence for any $\alpha_{x,y}$, there is a $z > y$ so that for any $w > z$, the factorization of $\alpha_{x,w}$ is the factorization of $\alpha_{x,z}$ followed by the factorization of $\alpha_{z,w}$. As $\mathcal{B}$ also does not have geometric character, it follows that a similar statement holds for the maps $\beta_{x,y}$.

Consequently, we can choose the sequences $\{m_i\}$ and $\{n_i\}$ so that the factorization of α_{m_j,m_k} is the factorizations of $\alpha_{m_i,m_{i+1}}$ for $j \leq i < k$ in order and similarly for the β_{n_j,n_k}.

Since

$$\beta_{n_k,n_{k+1}} \circ \beta_{n_{k-1},n_k} = (\phi_{2k+1} \circ \phi_{2k}) \circ (\phi_{2k-1} \circ \phi_{2k-2})$$

$$\alpha_{m_k,m_{k+1}} \circ \alpha_{m_{k-1},m_k} = (\phi_{2k+2} \circ \phi_{2k+1}) \circ (\phi_{2k} \circ \phi_{2k-1}),$$

it follows that the unique factorization of $\phi_k \circ \phi_{k+1}$ is the unique factorization of ϕ_k followed by that of ϕ_{k+1} for every $k > 1$. Since $\alpha_{m_k,m_{k+1}} = \phi_{2k+2} \circ \phi_{2k+1}$ and $\beta_{n_k,n_{k+1}} = \phi_{2k+1} \circ \phi_{2k}$ for every $k > 1$, it follows that (after removing all 1-tuples) the sequences of tuples in the presentations of $\mathcal{A}$ and $\mathcal{B}$ have the same unique factorization, except for possibly different initial segments (α_{1,m_1} and β_{1,n_1}).

This completes case 2 and the classification. □

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